Almost Affine Lambda Terms

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Makoto Kanazawa
National Institute of Informatics, Tokyo, Japan

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Jaśkowski, S. 1963.
Über Tautologien, in welchen keine Variable mehr als zweimal vorkommt.
Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **9** (12–15), 219–228.

ÜBER TAUTOLOGIEN, IN WELCHEN KEINE VARIABLE MEHR ALS ZWEIMAL VORKOMMT

von S. Jaśkowski in Torun (Polen)

H. Thiele hat im Jahre 1960 das Problem gestellt, das implikative Teilsystem des Aussagenkalküls mit den Axiomen

$$CCpqCCqrCpr, \quad CCpCqrCqCpr, \quad CpCqp$$

zu untersuchen. Hier wird für dieses System[1]) und für ein anderes, in dem das letzte Axiom durch ein schwächeres, nämlich durch Cpp ersetzt wird, ein Entscheidungsverfahren angegeben. Die Methode beruht auf einer Untersuchung von gewissen allgemeinen Eigenschaften der Ausdrücke, in welchen keine Satzvariable mehr als zweimal vorkommt. Dabei wird eine dreiwertige Matrix benutzt. Das Problem der charakteristischen Matrizen für die oben erwähnten Systeme ist bisher nicht gelöst.

1. Ein Lemma über Äquivalenzrelationen

Reflexive, symmetrische und transitive Relationen auf einer Menge M heißen Äquivalenzrelationen über M. Es seien $\sim_1, \sim_2$ Äquivalenzrelationen über der Menge M. Es ist bekannt, daß es über M eine eindeutig bestimmte Kompositionsäquivalenz $\sim_0$ von $\sim_1, \sim_2$ mit folgenden Eigenschaften gibt:

On tautologies, in which no [propositional] variable occurs more than twice.

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Hirokawa, S. 1992.
Balanced formulas, BCK-minimal formulas and their proofs.
Logical Foundations of Computer Science — Tver '92, pp. 198–208.

Here a formula is balanced iff no type variable occurs more than twice. Balanced formulas are called as formulas with one-two-property in [6, 12]. We denote the set of balanced formulas as $F_{1,2}$. The author learned a result by Jaskowski [12] from P. Idzjak which states that

$$BCK \cap F_{1,2} = LJ \cap F_{1,2}.$$

By analysis of the type assignment figures, we prove that if a λ-term in β-normal form has balanced type, then it is a BCK-λ-term. This gives a direct proof for Jaskowski's result.

A balanced sequent provable in intuitionistic logic has an *affine* inhabitant.

Hirokawa slightly improved Jaskowski's result.

λ-terms

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x	variable
λx.M	abstraction
MN	application

Affine λ-terms

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x	variable	
λx.M	abstraction	
MN	application	$FV(M) \cap FV(N) = \varnothing$

Affine λ-terms are *typable* (Hindley 1989).

Every variable is used at most once. Interesting properties of affine lambda terms have to do with their typings.

Type assignment to λ-terms

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$$x_1{:}\alpha_1,\ldots,x_n{:}\alpha_n \Rightarrow M{:}\alpha$$

type environment

$$x{:}\alpha \Rightarrow x{:}\alpha$$

$$\frac{\Gamma \Rightarrow M{:}\beta}{\Gamma - \{x{:}\alpha\} \Rightarrow \lambda x.M{:}\alpha \to \beta}\to I \qquad \frac{\Gamma \Rightarrow M{:}\alpha \to \beta \quad \Delta \Rightarrow N{:}\alpha}{\Gamma \cup \Delta \Rightarrow MN{:}\beta}\to E$$

Type environment includes type declarations for all and only variables that occur free in the term.

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$\Gamma \Rightarrow \alpha$

sequent

- $\vdash \Gamma \Rightarrow M{:}\alpha$
- $\Gamma \Rightarrow \alpha$ is a typing of M
- M is an inhabitant of $\Gamma \Rightarrow \alpha$

Some terminology.

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$\Gamma \Rightarrow \alpha$ is a principal typing of M

⇕

$\vdash \Gamma' \Rightarrow M{:}\alpha'$ just in case $(\Gamma' \Rightarrow \alpha') = (\Gamma \Rightarrow \alpha)\theta$ for some type substitution θ

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Polarity of type occurrences

$$x{:}(p^{-} \to q^{+}) \to r^{+} \to s^{-},\ y{:}p^{+} \to q^{-} \Rightarrow r^{-} \to s^{+}$$

$\Gamma \Rightarrow \alpha$ is balanced

⇕

each atomic type has at most 1 positive and at most 1 negative occurrence in $\Gamma \Rightarrow \alpha$

This stricter definition of "balanced" is from Mints' textbook on intuitionistic logic and is more convenient for my purposes.

Affine λ-terms and balanced sequents

- Principal typing of an affine λ-term is balanced (Belnap 1976, Hirokawa 1992).
- A balanced sequent has at most one inhabitant up to βη-equality (Coherence Theorem, Mints 1981, Babaev and Solov'ev 1982).
- A β-normal inhabitant of a balanced sequent is always affine (Jaśkowski 1963, Hirokawa 1992).

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These properties extend to what I call "almost affine" lambda terms.

Almost affine λ-terms

$$x{:}\alpha \Rightarrow x{:}\alpha$$

$$\frac{\Gamma \Rightarrow M{:}\beta}{\Gamma - \{x{:}\alpha\} \Rightarrow \lambda x.M{:}\alpha \to \beta}\;{\to}\mathrm{I} \qquad \frac{\Gamma \Rightarrow M{:}\alpha \to \beta \quad \Delta \Rightarrow N{:}\alpha}{\Gamma \cup \Delta \Rightarrow MN{:}\beta}\;{\to}\mathrm{E}$$

$$\mathrm{ran}(\Gamma \cap \Delta) \subseteq \mathrm{At}$$

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The definition of almost affine lambda-terms refer to their typing.

$$\dfrac{\dfrac{\dfrac{\dfrac{y{:}q \to q \to p \quad x{:}q}{yx{:}q \to p}\;{\to}\mathrm{E} \quad x{:}q}{yxx{:}p}\;{\to}\mathrm{E}}{\lambda x.yxx{:}q \to p}\;{\to}\mathrm{I} \quad \dfrac{z{:}r \to q \quad w{:}r}{zw{:}q}\;{\to}\mathrm{E}}{(\lambda x.yxx)(zw){:}p}\;{\to}\mathrm{E}$$

$$\dfrac{\dfrac{y{:}q \to q \to p \quad \dfrac{z{:}r \to q \quad w{:}r}{zw{:}q}\;{\to}\mathrm{E}}{y(zw){:}q \to p}\;{\to}\mathrm{E} \quad \dfrac{z{:}r \to q \quad w{:}r}{zw{:}q}\;{\to}\mathrm{E}}{y(zw)(zw){:}p}\;{\to}\mathrm{E}$$

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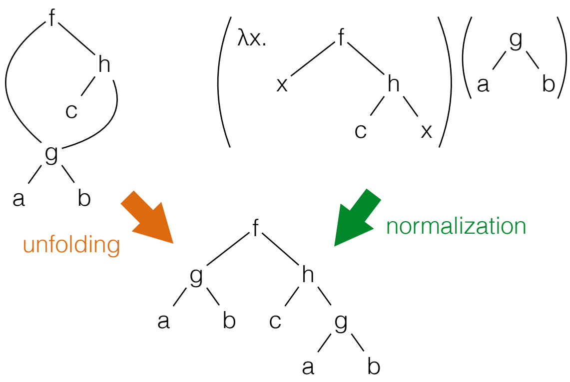
An example of an almost affine lambda term.

Its beta-normal form is not almost affine.

Term graph (tree with sharing)



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Almost affine lambda terms may be thought of as a generalization of “term graphs”.

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$\overset{+}{y}{:}\overset{+}{q} \rightarrow \overset{-}{q} \rightarrow p,\ z{:}\overset{+}{r} \rightarrow \overset{-}{q},\ w{:}\overset{-}{r} \Rightarrow (\lambda x.yxx)(zw){:}\overset{+}{p}$

$\Gamma \Rightarrow \alpha$ is negatively non-duplicated

⇕

each atomic type has at most one negative occurrence in $\Gamma \Rightarrow \alpha$

Almost affine λ-terms and negatively non-duplicated sequents

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- Principal typing of an almost affine λ-term is negatively non-duplicated (Aoto 1999, Kanazawa 2007, 2011).
- A negatively non-duplicated sequent has at most one inhabitant up to βη-equality (Aoto and Ono 1994).
- An inhabitant of a negatively non-duplicated sequent is always βη-equal to an almost affine λ-term.

The coherence theorem about affine lambda terms and the correspondence between affine lambda terms and balanced sequents extend to almost affine lambda terms and negatively non-duplicated sequents.

This paper proves the third bullet point.

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Affine λ-terms and balanced sequents

- Principal typing of an affine λ-term is balanced.
- A balanced sequent has at most one inhabitant up to βη-equality.
- A β-normal inhabitant of a balanced sequent is always affine.

Affine vs. balanced.

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Almost affine λ-terms and negatively non-duplicated sequents

- Principal typing of an almost affine λ-term is negatively non-duplicated.
- A negatively non-duplicated sequent has at most one inhabitant up to βη-equality.
- An inhabitant of a negatively non-duplicated sequent is always βη-equal to an almost affine λ-term.

Almost affine vs. negatively non-duplicated.
The third property is stated slightly differently due to the fact that the class of almost affine lambda-terms is not closed under beta-reduction.

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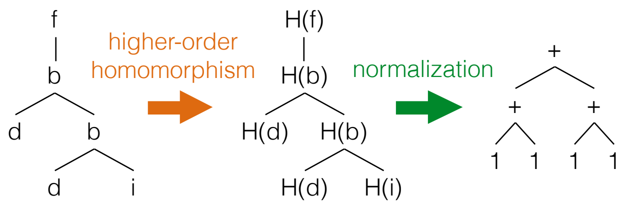
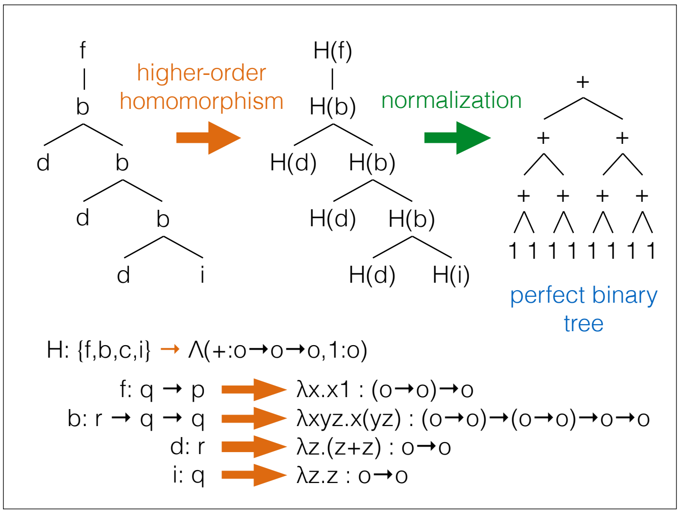
Almost affine λ-terms and tree transductions



H: {f,b,c,i} → Λ(+:o→o→o,1:o)

f: q → p ➡ λx.x1 : (o→o)→o
b: r → q → q ➡ λxyz.x(yz) : (o→o)→(o→o)→o→o
d: r ➡ λz.(z+z) : o→o
i: q ➡ λz.z : o→o

Why woud one be interested in the question of what lambda terms inhabit negatively non-duplicated sequents?
Almost affine lambda terms characterize an important class of tree transformations.



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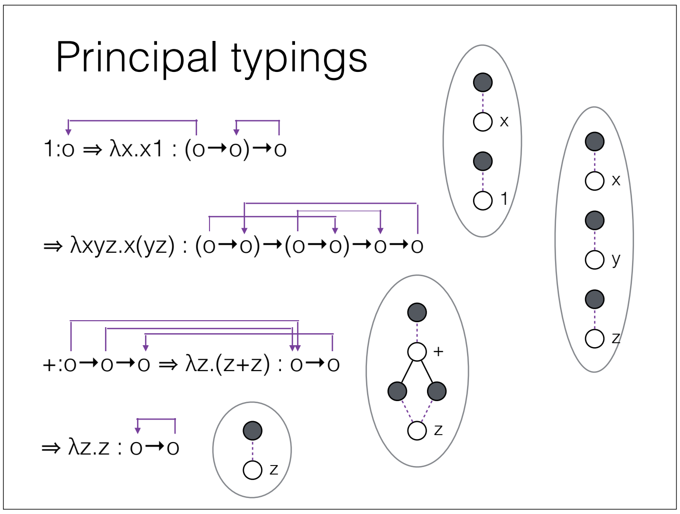
With almost affine lambda terms, normalization is easy to carry out (a kind of normalization by evaluation) using principal typings.



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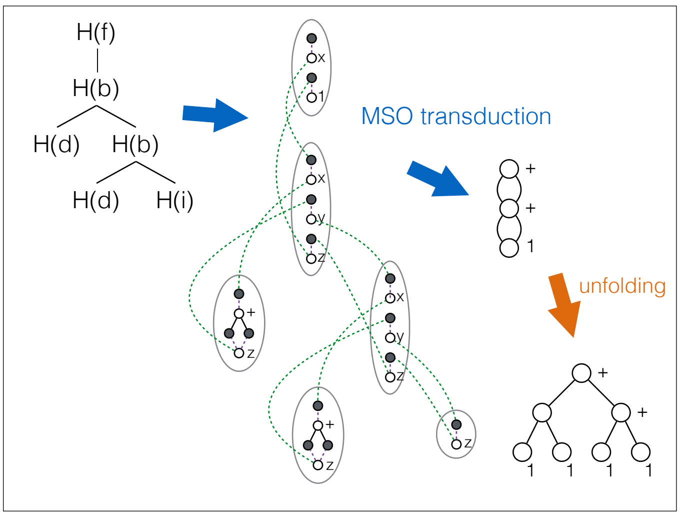
Graph representations of principal typings.



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We can compute the normal form using these graphs.

The output graph is defined in terms of relations on the input graph defined by MSO formulas.

Tree transductions

Almost affine higher-order homomorphism + normalization ≡ MSO transduction + unfolding

Kanazawa 2009

- Principal typing as graph
- Principal typing of an almost affine λ-term is negatively non-duplicated

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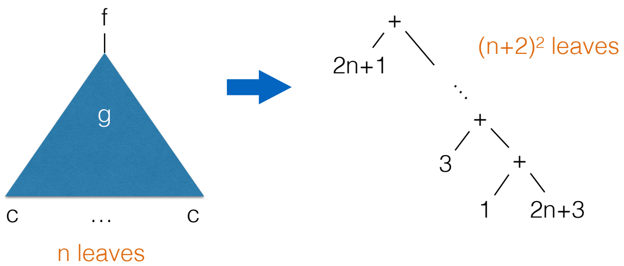
Correspondence.

One direction holds for all lambda terms that have negatively non-duplicated principal typings.

A more complex example



n abbreviates (⋯(1+1)+⋯)+1

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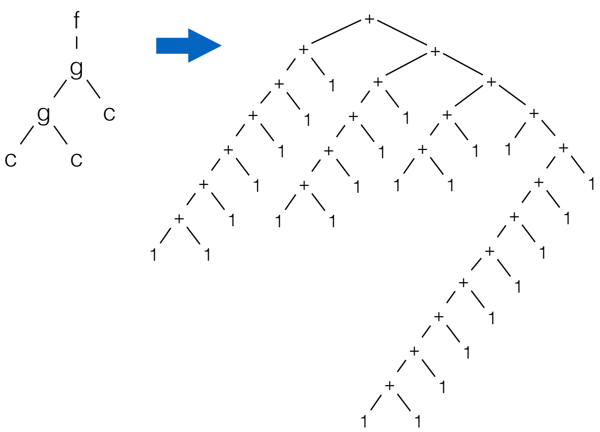

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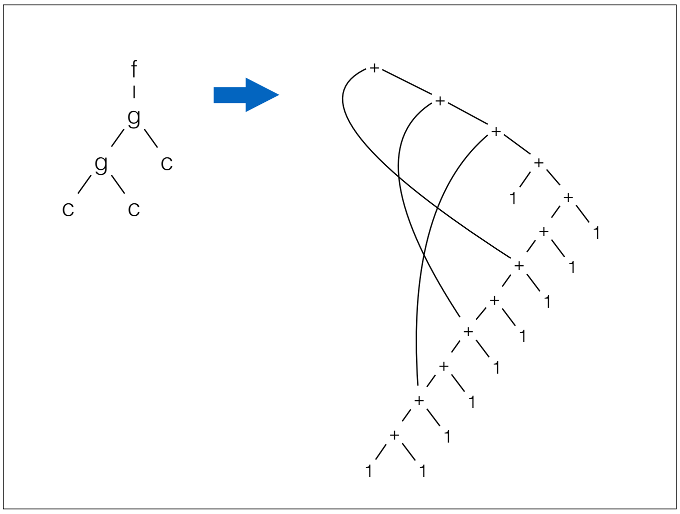

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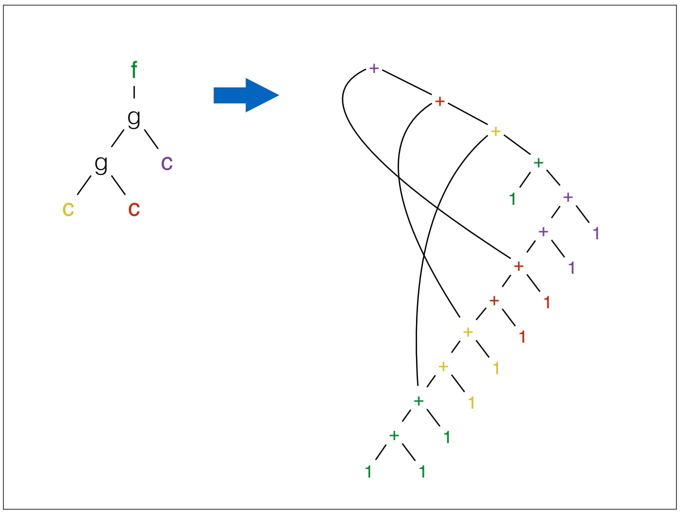

This term graph can be obtained by MSO transduction.

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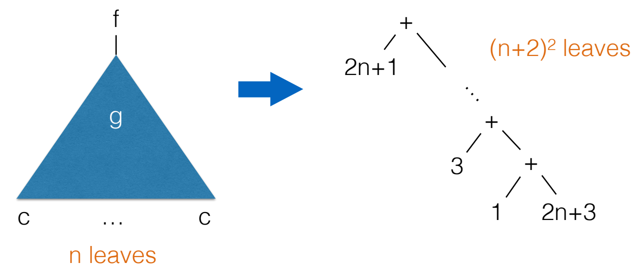

Higher-order homomorphism

$f := \lambda x.x((1+1)+1)(\lambda sy.y(1+s)(\lambda t.t))$

$g := \lambda x_1 x_2 iw.x_1 i(\lambda s_1 y_1.x_2 s_1(\lambda s_2 y_2.ws_2(\lambda jv.y_1 j(\lambda t_1.y_2 t_1(\lambda t_2.vt_2)))))$

$c := \lambda iw.w((i+1)+1)(\lambda jv.v(i+j))$

So there's a corresponding almost affine higher-order homomorphism.

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Almost affine λ-terms and negatively non-duplicated sequents

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- A negatively non-duplicated sequent has at most one inhabitant up to βη-equality (Aoto and Ono 1994).
- An inhabitant of a negatively non-duplicated sequent is always βη-equal to an almost affine λ-term.

MSO transduction is obtained only for almost affine higher-order homomorphism.

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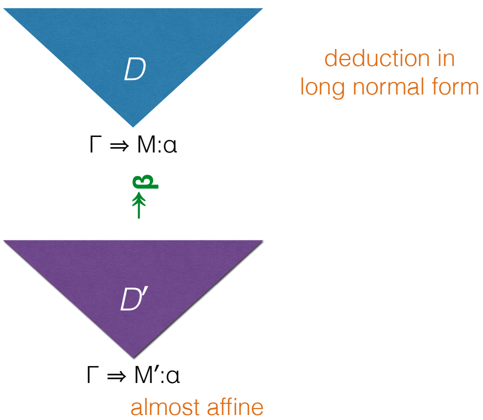
$\Gamma \Rightarrow \alpha$ negatively non-duplicated



Some idea of how the theorem is proved.

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$$\frac{y:\beta_1\to\cdots\to\beta_n\to p \quad \Gamma_1 \Rightarrow M_1:\beta_1 \quad \dots \quad \Gamma_n \Rightarrow M_n:\beta_n}{y:\beta_1\to\cdots\to\beta_n\to p,\ \Gamma_1\cup\cdots\cup\Gamma_n \Rightarrow yM_1\ M_n:p}\ \to E$$

$\{ x:\delta \in \Gamma \mid \delta \notin At,\ x:\delta \in \Gamma_i \cap \Gamma_j,\ i \neq j \} = \{y_1:\delta_1,\dots,y_m:\delta_m\} \neq \varnothing$

$tail(\delta_i)=q_i$

Find a k such that
$M \equiv_\alpha N[z:=y_k P_1 \dots P_l]$ and
$x \in FV(N) \cap FV(y_k P_1 \dots P_l)$ implies $\Gamma(x) \in At$

Problematic case.
Try to extract an atomic typed subdeduction.

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Find a k such that
$M \equiv_\alpha N[z{:=}y_k P_1 \ldots P_l]$ and
$x \in FV(N) \cap FV(y_k P_1 \ldots P_l)$ implies $\Gamma(x) \in At$

E

$$\dfrac{\dfrac{\Gamma', z{:}q_k \Rightarrow N{:}p}{\Gamma' \Rightarrow \lambda z.N{:}q_k \to p}\,{\to}\mathrm{I} \qquad \overset{F}{\Gamma'' \Rightarrow y_k P_1 \ldots P_l{:}q_k}}{\Gamma' \cup \Gamma'' \Rightarrow (\lambda z.N)(y_k P_1 \ldots P_l){:}p}\,{\to}\mathrm{E}$$

$\downarrow_\beta$

D

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$\alpha = \beta_1 \to \cdots \to \beta_i \to \cdots \to \beta_n \to p$

↑ negative subpremise of α

- If $\gamma = \beta_i$ or γ is a positive subpremise of β_i, then γ is a negative subpremise of α.
- If γ is a negative subpremise of β_i, then γ is a positive subpremise of α.

Important lemma concerning the negative subpremise property.

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$\Gamma = x_1{:}\alpha_1, \ldots, x_n{:}\alpha_n$

γ is a positive/negative subpremise of $\Gamma \Rightarrow \alpha$

if

γ is a positive/negative subpremise of $\alpha_1 \to \cdots \to \alpha_n \to \alpha$

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Definition

$\Gamma \Rightarrow \alpha$ has negative subpremise property

⟺ for all negative subpremises γ, δ of $\Gamma \Rightarrow \alpha$, $\mathrm{tail}(\gamma) = \mathrm{tail}(\delta)$ implies $\gamma = \delta$

negatively non-duplicated

⇓

negative subpremise property

$((p \to \bar{q}) \to r) \to s,\ p \to \bar{q},\ (p \to q) \to r \Rightarrow s$

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Lemma

$\Gamma \Rightarrow \alpha$, $\Gamma' \Rightarrow \alpha$ negatively non-duplicated

$\Gamma \cup \Gamma'$ injective

$\Gamma \cup \Gamma' \Rightarrow \alpha$ has negative subpremise property

$\vdash \Gamma \Rightarrow M{:}\alpha$

$\vdash \Gamma' \Rightarrow M'{:}\alpha$

⇓

$M =_{\beta\eta} M'$

This lemma implies Aoto and Ono's generalization of the coherence theorem.

The inductive proof is easy with "negative subpremise property".

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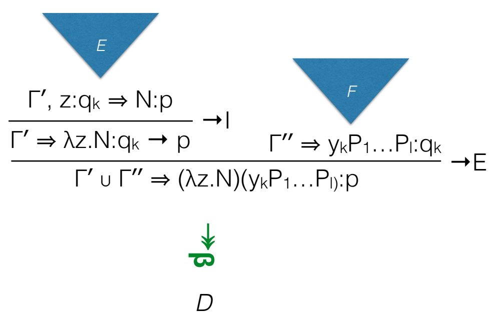
Find a k such that
$M \equiv_\alpha N[z{:=}y_k P_1 \ldots P_l]$ and
$x \in FV(N) \cap FV(y_k P_1 \ldots P_l)$ implies $\Gamma(x) \in At$



The lemma is used to find this k.

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