The Strong, Weak, and Very Weak Finite Context and Kernel Properties

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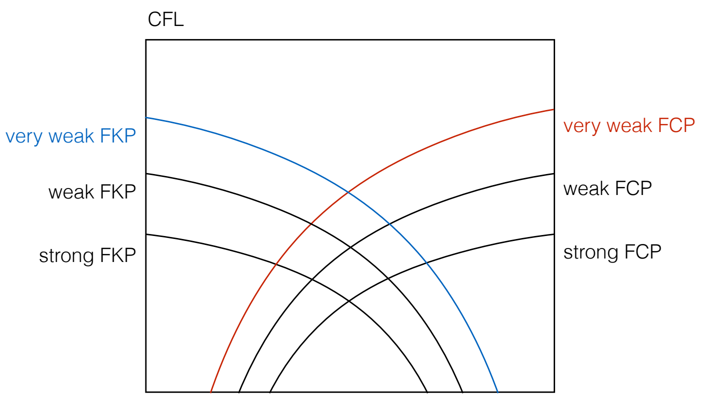


I'll spend most of my time filling you in on the background.
This talk is about six subclasses of the context-free languages, two of which turn out to be identical.
The classification has to do with work on distributional learning of context-free languages.
I'll first introduce the strongest of the two types of properties of CFGs.

Strings and Contexts

$L \subseteq \Sigma^*$: fixed language

The "acceptance" relation $R_L \subseteq \Sigma^* \times (\Sigma^* \times \Sigma^*)$ between strings and contexts defined by

$$x\ R_L\ (u,v) \Leftrightarrow uxv \in L$$

induces a **Galois connection**

$$(\mathcal{P}(\Sigma^*), \subseteq) \longleftrightarrow (\mathcal{P}(\Sigma^* \times \Sigma^*), \supseteq)$$

$$X \subseteq \Sigma^* \mapsto X^{\triangleright} = \{ (u,v) \mid uXv \subseteq L \}$$

$$W^{\triangleleft} = \{ x \mid uxv \in L \text{ for all } (u,v) \in W \} \leftarrow W \subseteq \Sigma^* \times \Sigma^*$$

All of these properties are relative to a fixed language L, which is supposed to be the target of learning.
Pairs of strings are called "contexts".
R_L: the relation of a context "accepting" a string.

$(\cdot)^{\triangleright\triangleleft}$ is a closure operator.

- $X \subseteq X^{\triangleright\triangleleft}$
- $X_1 \subseteq X_2 \Rightarrow X_1^{\triangleright\triangleleft} \subseteq X_2^{\triangleright\triangleleft}$
- $X^{\triangleright\triangleleft\triangleright\triangleleft} = X^{\triangleright\triangleleft}$

$W^{\triangleleft}$ is a closed set (i.e., $W^{\triangleleft} = W^{\triangleleft\triangleright\triangleleft}$) for all $W \subseteq \Sigma^* \times \Sigma^*$.

L is a closed set.

$\because L = \{(\varepsilon,\varepsilon)\}^{\triangleleft}$

$(XY)^{\triangleright\triangleleft} = (X^{\triangleright\triangleleft}Y^{\triangleright\triangleleft})^{\triangleright\triangleleft}$

CFG

$G = (N, \Sigma, P, I)$

I: set of initial nonterminals

$L(G,A) = \{ x \in \Sigma^* \mid A \Rightarrow_G^* x \}$

$L(G) = \bigcup_{A \in I} L(G,A)$

I use a definition of CFGs slightly different from the standard one in that multiple initial nonterminals are allowed.

Strong FCP/FKP

A CFG $G = (N, \Sigma, P, I)$ has

- the **strong finite context property** if for each $A \in N$, there is some finite $C_A \subseteq \Sigma^* \times \Sigma^*$ such that

$$L(G,A) = C_A^{\triangleleft},$$

- the **strong finite kernel property** if for each $A \in N$, there is some finite $K_A \subseteq \Sigma^*$ such that

$$L(G,A) = K_A^{\triangleright\triangleleft}.$$

$\triangleright$, $\triangleleft$: relative to L(G)

The Dyck Language

$$S \to \varepsilon \mid aSbS$$

$$L(G,S) = D_1^* = \{(\varepsilon,\varepsilon)\}^{\triangleleft} = \{\varepsilon\}^{\triangleright\triangleleft}$$

I'll go over a few examples.
The standard grammar for the (one-sided) Dyck language satisfies both the strong FCP and the strong FKP.

Regular Sets

$M = (Q, \Sigma, \delta, q_I, F)$ DFA for L

$G = (Q, \Sigma, P, I)$

$$P = \{\, q \to ar \mid \delta(q,a) = r \,\} \cup \{\, q \to \varepsilon \mid q \in F \,\}$$

$$I = \{\, q_I \,\}$$

$$\hat{\delta}(q_I,u) = r \Rightarrow L(G,r) = \{\, x \mid \hat{\delta}(r,x) \in F \,\} = \{(u,\varepsilon)\}^{\triangleleft}$$

The right-linear grammar for a regular set corresponding to a DFA has the strong FCP.

Regular Sets

$\Sigma^*/{\equiv_L}$: syntactic monoid of L

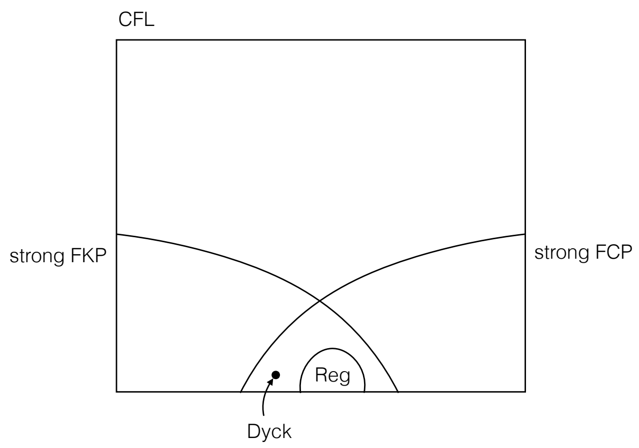
$G = (\Sigma^*/{\equiv_L}, \Sigma, P, I)$

$$P = \{\, [uv]_{\equiv_L} \to [u]_{\equiv_L}\, [v]_{\equiv_L} \mid u,v \in \Sigma^* \,\} \cup \{\, [a]_{\equiv_L} \to a \mid a \in \Sigma \cup \{\varepsilon\} \,\} \cup \{\, [u]_{\equiv_L} \to [v]_{\equiv_L} \mid [u]_{\equiv_L} \supseteq [v]_{\equiv_L} \,\}$$

$$I = \{\, [x]_{\equiv_L} \mid x \in L \,\}$$

$$L(G,[x]_{\equiv_L}) = [x]_{\equiv_L}^{\triangleright\triangleleft} = \{x\}^{\triangleright\triangleleft}$$

The CFG corresponding to the syntactic monoid of a regular set has the strong FKP.



$L = \{ w \in \{a,b\}^* \mid |w|_a \neq |w|_b \}$

Applying the pumping lemma to a long string in a^+, we get:

$S \Rightarrow^* a^i\ A\ a^j$,

$A \Rightarrow^+ a^k\ A\ a^l$ with $k+l > 0$,

$A \Rightarrow^* a^m$.

$$\{ a^{m+n(k+l)} \mid n \geq 0 \} \subseteq L(G,A).$$

$$a^{i+nk}\ L(G,A)\ a^{j+nl} \subseteq L \text{ for all } n \geq 0,$$

$$\{ b^{i+j+n(k+l)} \mid n \geq 0 \} \cap L(G,A) = \varnothing.$$

An example that satisfies neither the strong FCP nor the strong FKP.
L(G,A) includes an infinite subset of a^+, and there is an infinite set of contexts consisting of strings in a^+ that accept all strings in L(G,A).
The complement of L(G,A) includes an infinite subset of b^+.

$L = \{ w \in \{a,b\}^* \mid |w|_a \neq |w|_b \}$

- For any finite $K \subseteq \{a,b\}^*$, $K^{\triangleright\triangleleft} \cap a^*$ is finite.

  Let $m = \max \{ |w|_a - |w|_b \mid w \in K \}$.

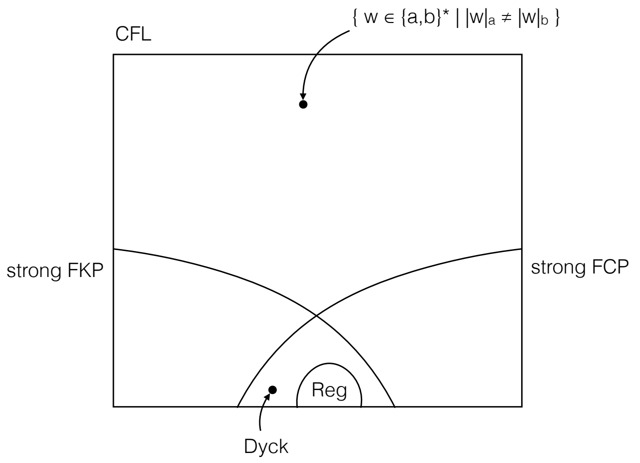
  $$K^{\triangleright} \supseteq \{ (\varepsilon, b^n) \mid n > m \},$$

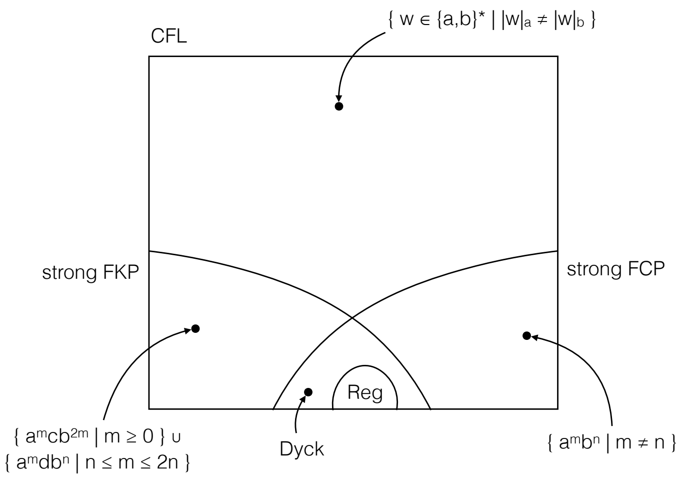
  $$K^{\triangleright\triangleleft} \cap a^* \subseteq \{ a^n \mid n \leq m \}.$$

  So $\{ a^{m+n(k+l)} \mid n \geq 0 \} \subseteq L(G,A) \neq K^{\triangleright\triangleleft}$.

- For any finite $C \subseteq \{a,b\}^* \times \{a,b\}^*$, $C^{\triangleleft} \cap b^*$ is co-finite.

  Since $\{ b^{i+j+n(k+l)} \mid n \geq 0 \} \cap L(G,A) = \varnothing$, $L(G,A) \neq C^{\triangleleft}$.





It's easy to see that the other two regions are also nonempty.

Learning from Positive Data and Membership Queries

positive presentation of L

$t_1, t_2, t_3, \ldots$ $\qquad$ $L = \{ t_n \mid n \geq 1 \}$

At stage n:

- positive data: $D_n = \{t_1,\ldots,t_n\}$
- membership queries

  $w \in L?$

  with $|w|$ bounded by a polynomial in the size of D_n

polynomial update time

The strong FCP and the strong FKP were introduced in the context of Alex Clark's work on "distributional learning" of context-free languages.

Dual Learner

- Assume the target language has a CFG with the strong FCP.
- Nonterminals: $C \subseteq \Sigma^* \times \Sigma^*$ with $1 \leq |C| \leq \mathbf{k}$.
- A production $C_0 \to w_0\, C_1\, w_1 \dots C_n\, w_n$ ($n \leq \mathbf{r}$, $w_j \in \Sigma^*$) is **valid** if

$$C_0^{\triangleleft} \supseteq w_0\, C_1^{\triangleleft}\, w_1 \dots C_n^{\triangleleft}\, w_n.$$

- This is approximated by

$$C_0^{\triangleleft} \supseteq w_0\, (E \cap C_1^{\triangleleft})\, w_1 \dots (E \cap C_n^{\triangleleft})\, w_n$$

where E is the set of substrings in the available positive data.

There are two learning algorithms, dual and primal.

Dual Learner

Parameters: positive integers **k**, **r**;
Data: positive presentation $t_1, t_2, t_3, \dots$ of L; membership oracle for L;

$D_0 := \varnothing$; $E_0 := \varnothing$; $J_0 := \varnothing$;
for i := 1, 2, 3,… **do**
  $D_i := D_{i-1} \cup \{t_i\}$; $E_i := \mathrm{Sub}(D_i)$;
  if $D_i \not\subseteq L(G_{i-1})$ **then**
    $J_i := \mathrm{Con}(D_i)$;
  else
    $J_i := J_{i-1}$;
  $N_i := \{ C \subseteq J_i \mid 1 \leq |C| \leq \mathbf{k} \}$;
  $P_i := \{ C_0 \to w_0\, C_1\, w_1 \dots C_n\, w_n \mid n \leq \mathbf{r}, C_j \in N_i, w_j \in E_i\ (j = 0,\dots,n),$
    $C_0^{\triangleleft} \supseteq w_0\, (E_i \cap C_1^{\triangleleft})\, w_1 \dots (E_i \cap C_n^{\triangleleft})\, w_n \}$;
  $I_i := \{ C \in N_i \mid E_i \cap C^{\triangleleft} \subseteq L \}$;
  output $G_i := (N_i, \Sigma, P_i, I_i)$;

$\mathrm{Sub}(D) = \{ x \mid uxv \in D \}$

$\mathrm{Con}(D) = \{ (u,v) \mid uxv \in D \}$

Why the Dual Learner Works

- If $A_0 \to w_0\, A_1\, w_1 \dots A_n\, w_n$ is a production of G and $L(G,A_j) = C_j^{\triangleleft}$, then the production $C_0 \to w_0\, C_1\, w_1 \dots C_n\, w_n$ is valid:

$$C_0^{\triangleleft} \supseteq w_0\, C_1^{\triangleleft}\, w_1 \dots C_n^{\triangleleft}\, w_n$$

- If A is an initial nonterminal of G and $L(G,A) = C^{\triangleleft}$, then $C^{\triangleleft} \subseteq L$.
- If every production of G_i is valid and every initial nonterminal C of G_i satisfies $C^{\triangleleft} \subseteq L$, then $L(G_i) \subseteq L$.
- If L = L(G) for some G with the strong FCP, the dual learner converges to some G′ such that L = L(G′).

Primal Learner

- Assume the target language has a CFG with the strong FKP.
- Nonterminals: $K \subseteq \Sigma^*$ with $1 \leq |K| \leq$ **k**.
- A production $K_0 \to w_0\, K_1\, w_1 \ldots K_n\, w_n$ ($n \leq$ **r**, $w_j \in \Sigma^*$) is **valid** if

$$K_0^{\rhd\lhd} \supseteq w_0\, K_1^{\rhd\lhd}\, w_1 \ldots K_n^{\rhd\lhd}\, w_n,$$

or, equivalently, if

$$K_0^{\rhd} \subseteq (w_0\, K_1\, w_1 \ldots K_n\, w_n)^{\rhd}.$$

- This is approximated by

$$J \cap K_0^{\rhd} \subseteq (w_0\, K_1\, w_1 \ldots K_n\, w_n)^{\rhd}$$

where J is the set of contexts contained in the available positive data.

Primal Learner

Parameters: positive integers **k**, **r**;
Data: positive presentation $t_1, t_2, t_3, \ldots$ of L; membership oracle for L;

$D_0 := \varnothing$; $E_0 := \varnothing$; $J_0 := \varnothing$;
for i := 1, 2, 3,… **do**
  $D_i := D_{i-1} \cup \{t_i\}$; $J_i := \mathrm{Con}(D_i)$;
  if $D_i \not\subseteq L(G_{i-1})$ **then**
    $E_i := \mathrm{Sub}(D_i)$;
  else
    $E_i := E_{i-1}$;
  $N_i := \{ K \subseteq E_i \mid 1 \leq |K| \leq \mathbf{k} \}$;
  $P_i := \{ K_0 \to w_0\, K_1\, w_1 \ldots K_n\, w_n \mid n \leq \mathbf{r},\ K_j \in N_i,\ w_j \in E_i\ (j = 0,\ldots,n),$
      $J_i \cap K_0^{\rhd} \subseteq (w_0\, K_1\, w_1 \ldots K_n\, w_n)^{\rhd} \}$;
  $I_i := \{ K \in N_i \mid K \subseteq L \}$;
  output $G_i := (N_i, \Sigma, P_i, I_i)$;

$\mathrm{Sub}(D) = \{ x \mid uxv \in D \}$

$\mathrm{Con}(D) = \{ (u,v) \mid uxv \in D \}$

Why the Primal Learner Works

- If $A_0 \to w_0\, A_1\, w_1 \ldots A_n\, w_n$ is a production of G and $L(G,A_j) = K_j^{\rhd\lhd}$, then the production $K_0 \to w_0\, K_1\, w_1 \ldots K_n\, w_n$ is valid:

$$K_0^{\rhd\lhd} \supseteq w_0\, K_1^{\rhd\lhd}\, w_1 \ldots K_n^{\rhd\lhd}\, w_n$$

- If A is an initial nonterminal of G and $L(G,A) = K^{\rhd\lhd}$, then $K \subseteq L$.
- If every production of G_i is valid and every initial nonterminal K of G_i satisfies $K \subseteq L$, then $L(G_i) \subseteq L$.
- If $L = L(G)$ for some G with the strong FKP, the primal learner converges to some G′ such that $L = L(G')$.

Weak FCP/FKP

A CFG $G = (N, \Sigma, P, I)$ has

- the **weak finite context property** if for each $A \in N$, there is some finite $C_A \subseteq \Sigma^* \times \Sigma^*$ such that

$$L(G,A)^{\triangleright\triangleleft} = C_A^{\triangleleft},$$

- the **weak finite kernel property** if for each $A \in N$, there is some finite $K_A \subseteq \Sigma^*$ such that

$$L(G,A)^{\triangleright\triangleleft} = K_A^{\triangleright\triangleleft}.$$

$\triangleright$, $\triangleleft$: relative to $L(G)$

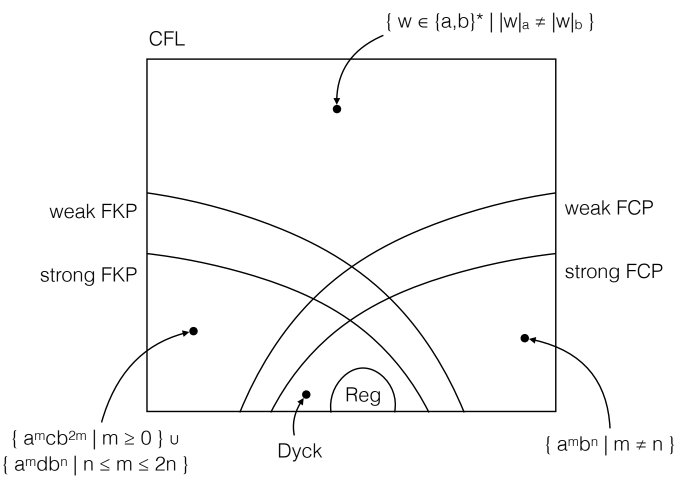
The strong FCP/FKP were soon found to be stronger than necessary.

Weak FCP/FKP Is Sufficient

- $X_0 \supseteq w_0 X_1 w_1 \dots X_n w_n$ implies

$$X_0^{\triangleright\triangleleft} \supseteq w_0 X_1^{\triangleright\triangleleft} w_1 \dots X_n^{\triangleright\triangleleft} w_n.$$

- $X \subseteq L$ implies $X^{\triangleright\triangleleft} \subseteq L$.
- The dual learner converges to a correct grammar if L has a grammar with the weak FCP.
- The primal learner converges to a correct grammar if L has a grammar with the weak FKP.



Strong and Weak FCP/FKP, Dual and Primal Learners

Ryo Yoshinaka Alex Clark

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I’ve covered the background to this work, due to these people.

Yoshinaka’s Question

- Is there a context-free language that has a grammar satisfying the weak FCP/FKP but has no grammar satisfying the strong FCP/FKP?

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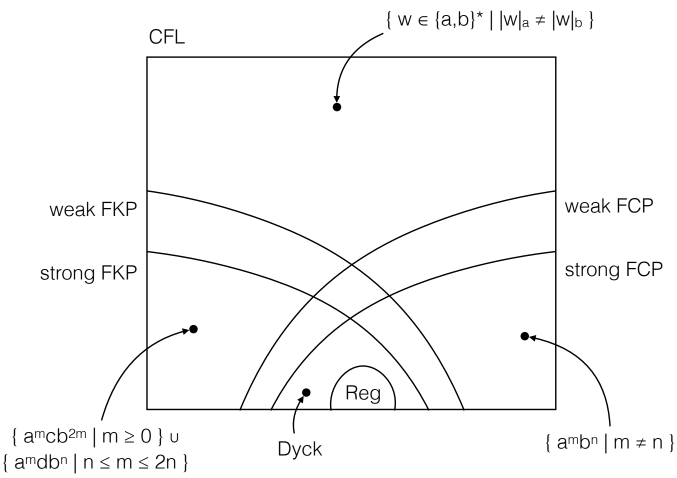

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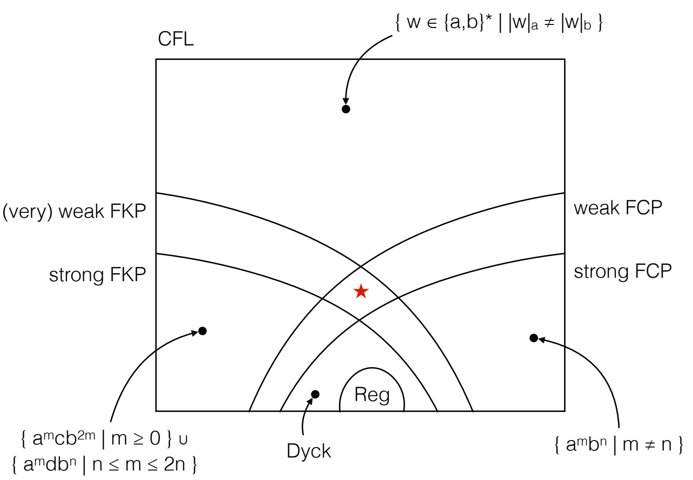
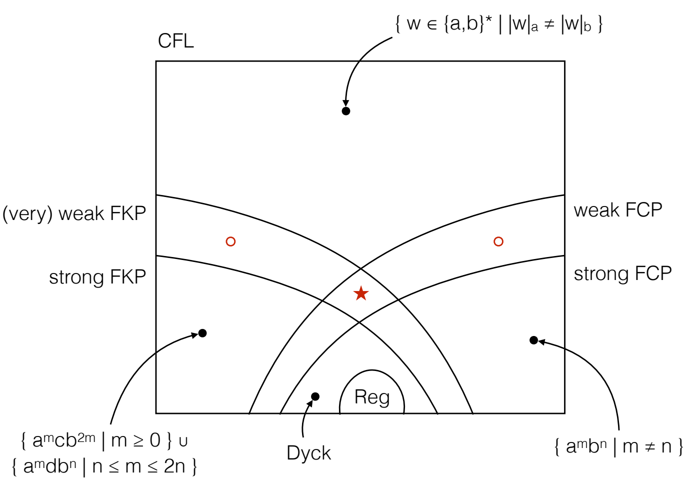
$$L = L_1 \cup L_2 \cup L_3,$$
$$L_1 = \{\, w_1 \# w_2 \# \ldots \# w_n \$ w_n^R \ldots w_2^R w_1^R \mid n \geq 1, w_1, \ldots, w_n \in \{a,b\}^* \,\},$$
$$L_2 = \{\, w y c^i d^i e^j z \mid w, z \in \{a,b\}^*, y \in (\#\{a,b\}^*)^*, i, j \geq 0, |w|_a \geq |w|_b \,\},$$
$$L_3 = \{\, w y c^i d^j e^j z \mid w, z \in \{a,b\}^*, y \in (\#\{a,b\}^*)^*, i, j \geq 0, |w|_a \leq |w|_b \,\}.$$

- L has a grammar satisfying both the weak FCP and the weak FKP.
- L has no grammar satisfying the strong FCP or the strong FKP.
- Every CFG G for L has a nonterminal A such that L(G,A) is not a closed set.

$$L(G,A)^{\triangleright\triangleleft} \cap c^*d^*e^* = \{\, c^i d^i e^i \mid i \geq 0 \,\}$$

If $c^i d^j e^k$ is accepted in all contexts that accept v\$z, then i = j = k.





The nonemptiness of two regions sometimes implies the nonemptiness of another region.
The observation that the weak FCP/ weak FKP is sufficient was meaningful.

More Fundamental Questions

- Does the weak FCP/FKP exactly capture the languages that the dual/primal learner successfully learns?
- Does the grammar that the dual/primal learner converge to always satisfy the weak FCP/FKP?

Are the weak FCP and the weak FKP weak enough?
Turns out it's easy to define properties that exactly capture the dual/primal learners.

Pre-fixed Points

A sequence $(X_A)_{A\in N}$ of subsets of Σ^* is a **pre-fixed point** of $G = (N, \Sigma, P, I)$ if for each production $A_0 \to w_0 A_1 w_1 \dots A_n w_n$ in P,

$$X_{A_0} \supseteq w_0 X_{A_1} w_1 \dots X_{A_n} w_n.$$

- $(L(G,A))_{A\in N}$ is the least pre-fixed point of G.
- $(L(G,A)^{\triangleright\triangleleft})_{A\in N}$ is the least pre-fixed point of G consisting of closed sets.

Very Weak FCP

A CFG $G = (N, \Sigma, P, I)$ has the **very weak finite context property** if there is a sequence $(C_A)_{A\in N}$ of finite subsets of $\Sigma^* \times \Sigma^*$ such that

- $(C_A^{\triangleleft})_{A\in N}$ is a pre-fixed point of G,
- $\bigcup_{A\in I} C_A^{\triangleleft} \subseteq L(G)$.

strong FCP	$L(G,A) = C_A^{\lhd}$	$\bigcup_{A\in I} L(G,A) \subseteq L(G)$
weak FCP	$L(G,A)^{\rhd\lhd} = C_A^{\lhd}$	$\bigcup_{A\in I} L(G,A)^{\rhd\lhd} \subseteq L(G)$
very weak FCP	$(C_A^{\lhd})_{A\in N}$ is a pre-fixed point of G	$\bigcup_{A\in I} C_A^{\lhd} \subseteq L(G)$

Very Weak FCP Captures Dual Learner

- The dual learner converges to a correct grammar if L has a grammar with the **very** weak FCP.
- The grammar that the dual learner converges to always satisfies the **very** weak FCP.

Very Weak FKP

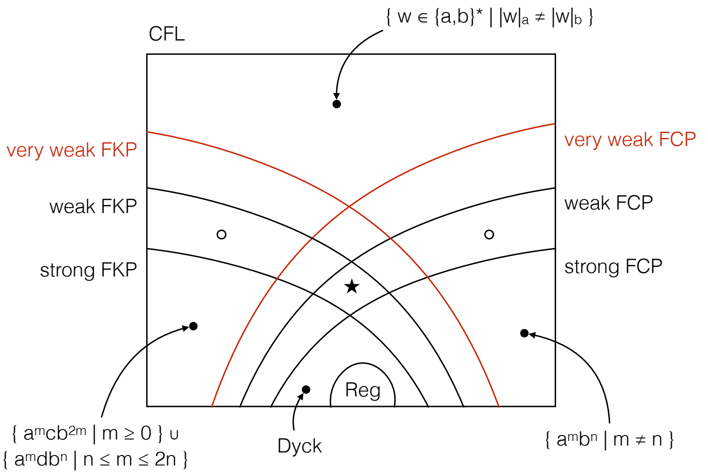
A CFG $G = (N, \Sigma, P, I)$ has the **very weak finite kernel property** if there is a sequence $(K_A)_{A\in N}$ of finite subsets of Σ^* such that

- $(K_A^{\rhd\lhd})_{A\in N}$ is a pre-fixed point of G,
- $\bigcup_{A\in I} K_A^{\rhd\lhd} \subseteq L(G)$ (or, equivalently, $\bigcup_{A\in I} K_A \subseteq L(G)$).

strong FKP	$L(G,A) = K_A^{\triangleright\triangleleft}$	$\bigcup_{A\in I} L(G,A) \subseteq L(G)$
weak FCP	$L(G,A)^{\triangleright\triangleleft} = K_A^{\triangleright\triangleleft}$	$\bigcup_{A\in I} L(G,A)^{\triangleright\triangleleft} \subseteq L(G)$
very weak FCP	$(K_A^{\triangleright\triangleleft})_{A\in N}$ is a pre-fixed point of G	$\bigcup_{A\in I} K_A^{\triangleright\triangleleft} \subseteq L(G)$

Very Weak FKP Captures Primal Learner

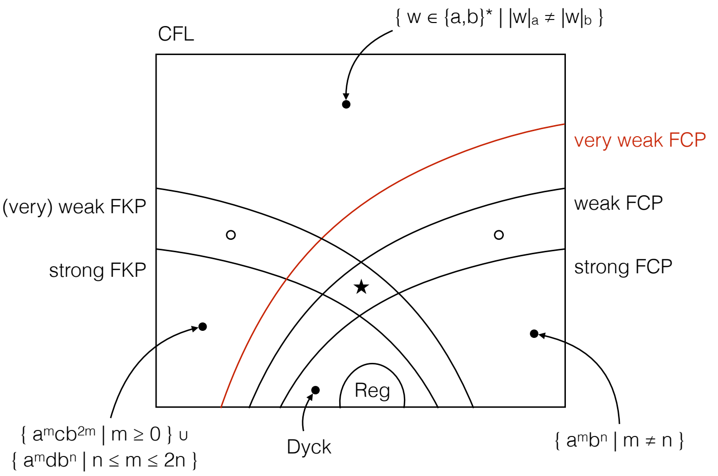
- The primal learner converges to a correct grammar if L has a grammar with the **very** weak FKP.
- The grammar that the primal learner converges to always satisfies the **very** weak FKP.

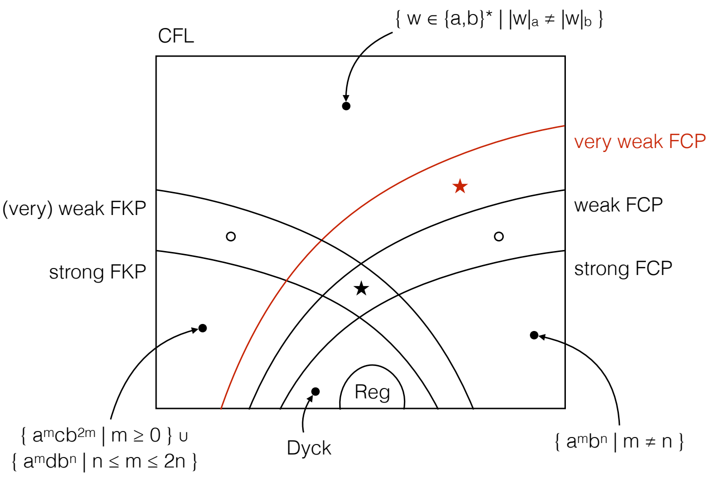


Everything inside one of the two red boundaries is polynomial-time learnable from positive data and membership queries.
Is this a real improvement?

Very Weak FKP ≡ Weak FKP

- L has a grammar with the very weak FKP if and only if L has a grammar with the weak FKP.
- The language that the primal learner converges to always satisfies the weak FKP.





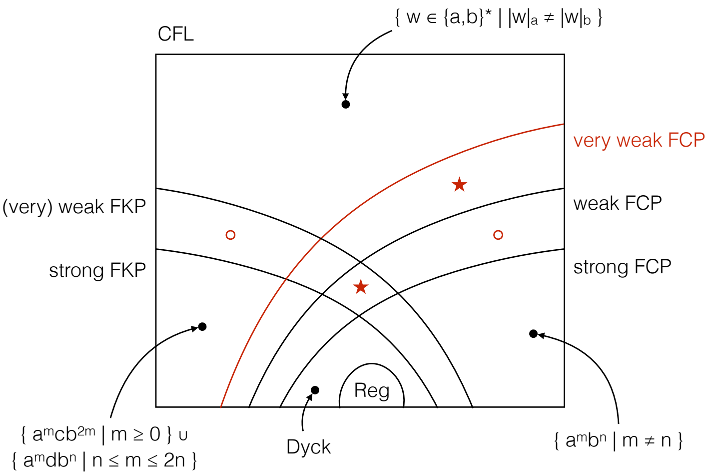
Main result of the paper.

$$L' = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5,$$

$$L_1 = \{\, w_1 \# w_2 \# \dots \# w_n \$ w_n^R \dots w_2^R w_1^R \mid n \geq 1, w_1, \dots, w_n \in \{a,b\}^* \,\},$$
$$L_2 = \{\, w y c^i d^i e^j z \mid w, z \in \{a,b\}^*, y \in (\#\{a,b\}^*)^*, i, j \geq 0, |w|_a \geq |w|_b \,\},$$
$$L_3 = \{\, w y c^i d^j e^j z \mid w, z \in \{a,b\}^*, y \in (\#\{a,b\}^*)^*, i, j \geq 0, |w|_a \leq |w|_b \,\}.$$
$$L_4 = \{\, v \$ z c^k \% f^l \mid v \in \{a,b,\#\}^*, z \in \{a,b\}^*, k, l \geq 0 \,\},$$
$$L_5 = \{\, v c^i d^i e^j z c^k \% f^l \mid v \in \{a,b,\#\}^*, z \in \{a,b\}^*, i, j, k, l \geq 0, j \neq k \,\},$$

- L′ has a CFG satisfying the very weak FCP, but has no CFG satisfying the weak FCP.

This example involves a phenomenon similar to what we saw with { w ∈ {a,b}* | $|w|_a \neq |w|_b$ }.



New results in red.

Conclusion

- The primal learner is exactly captured by the weak FKP.
- The dual learner is exactly captured by the very weak FCP, which is genuinely weaker than the weak FCP.