Symmetric quantifiers and the expressive power of binary determiners

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1 Introduction

It is generally admitted that in natural languages nominal unary determiners, that is functional expressions which form nouns phrases when applied to one common noun, denote not arbitrary type $\langle 1, 1\rangle$ quantifiers (relations between sets) but only those which satisfy the constraint of conservativity. This constraint informally says that to determine the truth conditions of sentences with such determiners one has not to take into account all sets determined by the arguments of the function, in particular the complement of the first set-theoretic argument does not matter. It follows from this that some type $\langle 1, 1\rangle$ are not "naturally" denotable by even complex unary determiners.

The conservativity fact is sometimes considered as a language universal: all types of determiners in all natural languages are conservative in the sense that they denote only conservative functions. Even if some non-conservative determiners are known it appears that they are rare and not arbitrary since they are systematically related to conservative determiners (cf. Zuber 2004a).

It has also been established that NLs display binary or even n-ary determiners that is functional expressions which form noun phrases from more than one common noun (cf. Keenan and Moss 1985). A simple example of such a binary determiner is given by the comparative determiner *more... than* as it occurs in the noun phrase *more students than teachers*.

Such n-ary determiners denote higher order quantifiers which are n-ary relations between sets or, equivalently, binary relations between relations and sets. The notion of conservativity, and other related notions discussed in the context of type $\langle 1, 1\rangle$ quantifiers easily generalise to higher order quantifiers denoted by n-ary determiners and consequently the universalistic claim concerning conservativity of n-ary determiners as well. Thus one considers as language universal the claim that n-ary, or more specifically binary, determiners in all NLs are conservative.

The purpose of this paper is to show that in some sense a stronger claim can be made concerning binary determiners: such determiners satisfy the natural constraint of symmetry. In the case of unary determiners symmetry just means that in simple sentences with such determiners one can permute verbal and nominal arguments without changing the truth-value of the whole. The notion of symmetry can be generalised to quantifiers denoted by binary (or even n-ary) determiners (Zuber 2007). It appears than that a huge majority of binary determiners denote symmetric quantifiers. For instance the comparative determiner *more...than* denotes a symmetric quantifier because in particular sentences in (1) are equivalent:

(1a) More students than teachers are Buddhists.
(1b) More Buddhists are students than teachers.

The paper is organised as follows. First, I will recall basic properties of "simple" type $\langle 1,1\rangle$ focusing on symmetric, and their duals, contrapositional quantifiers. It will be generally assumed that universe of discourse if finite. Then in the next section I will show how various properties, in particular conservativity and symmetry, extend to higher order quantifiers. Finally various known types of binary determiners will be examined in order to show that all of them are symmetric. All this will be done using the framework of Boolean semantics (Keenan and Faltz 1986) since, as it will be shown, various involved classes of quantifiers have the Boolean structure. This fact will be used to make more precise some formal claims.

2 Variety of simple quantifiers

In this section we are interested in the denotations of (unary) nominal determiners. These are expressions (like *every, no, some...including Lea, most*) which combine with common nouns to form noun phrases. Thus, semantically, they are functions from $P(E)$ onto type $\langle 1\rangle$ quantifiers, where E is the universe of discourse and a type $\langle 1\rangle$ quantifier is a set of sub-sets of E. They are type $\langle 1,1\rangle$ quantifiers and will be called here *simple quantifiers*. These quantifiers can be viewed as binary relations on sets. Indeed a type $\langle 1,1\rangle$ quantifier F, which is a function in $[P(E) \rightarrow [P(E) \rightarrow \{0,1\}]]$ corresponds to the binary relation Q between sets defined by $QXY \Leftrightarrow F(X)(Y) = 1$. The set of all type $\langle 1,1\rangle$ quantifiers, or the set of unrestricted functions belonging to $[P(E) \rightarrow [P(E) \rightarrow \{0,1\}]]$ will be denoted by $PDET$. This set forms an atomic Boolean algebra. Any member F of $PDET$ has a Boolean complement $\neg F$ and a post-complement $F\neg$ defined in the usual way.

It has been noticed that the class $PDET$ is too "large" to be the set of possible denotations for unary determiners since all functions denoted by such determiners satisfy various constraints. One of the best known such constraints on possible denotations of determiners is conservativity. By definition:

D1: F is conservative or $F \in CONS$ iff for any property X, Y and Z if $X \cap Y = X \cap Z$ then $F(X)(Y) = F(X)(Z)$

Conservativity of type $\langle 1,1\rangle$ quantifiers can additionally be formulated in two different ways:

Fact 1 (cf. Keenan and Faltz 1986) $F \in CONS$ iff for any property X, Y one has $F(X)(Y) = F(X)(X \cap Y)$
Fact 2 (Zuber 2005): $F \in CONS$ iff for any property X, Y one has $F(X)(Y) = F(X)(X' \cup Y)$

The set $CONS$ forms an atomic Boolean algebra. Atoms of $CONS$ are defined as follows (Keenan 1993):

Fact 3: For any $A, B \subseteq E$ the function $F_{A,B}$ such that $F_{A,B}(X)(Y) = 1$ iff $X = A$

and $X \cap Y = B$.

The constraint of conservativity considerable reduces the number of conservative functions in comparison with the number of unrestricted functions. Thus we have (Thijsse 1985):

Proposition 4: If $|E| = n$ then $|PDET| = 2^{4^n}$ and $|CONS| = 2^{3^n}$

It follows from Proposition 4 that in the universe with just two elements we have 65,536 of unrestricted type $\langle 1, 1\rangle$ quantifiers among which there are only 512 conservative ones.

There are various empirically and theoretically important sub-classes of the algebra $CONS$. Thus $CONS$ has two sub-algebras, the algebra INT of intersective functions, and the algebra $CO\text{-}INT$ of co-intersective functions (Keenan 1993). By definition:

D2: $F \in INT$, iff for all properties X, Y, Z and W, if $X \cap Y = Z \cap W$ then $F(X)(Y) = F(Z)(W)$.
D3: $F \in CO\text{-}INT1$ iff for all properties X, Y, Z and W, if $X - Y = Z - W$ then $F(X)(Y) = F(Z)(W)$.

Intersective and co-intersective type $\langle 1, 1|rangle$ quantifiers can be defined in four equivalent ways as shown by the following facts (Zuber 2007):

Fact 5: The following four conditions are equivalent: (i) $F \in INT$, (ii) $F(X)(Y) = F(X \cap Y)(X \cap Y)$, (iii) $F(X)(Y) = F(E)(X \cap Y)$, (iv) $F(X)(Y) = F(X \cap Y)(E)$
Fact 6: The following four conditions are equivalent: $F \in CO\text{-}INT1$, (ii) $F(X)(Y) = F(X - Y)(X' \cup Y)$, (iii) $F(X)(Y) = F(X - Y)(\emptyset)$, (iv) $F(X)(Y) = F(E)(X' \cup Y)$

Both sets, INT and $CO\text{-}INT$, form atomic (and complete) Boolean algebras. Their atoms are determined by a property (Keenan 1993): for any property P the function F_P such that $F_P(X)(Y) = 1$ iff $X \cap Y = P$ is a atom of INT and the function F_P such that $F_P(X)(Y) = 1$ iff $X - Y = P$ is an atom of $CO\text{-}INT$. Exclusion determiners (Zuber 1998) denote such atomic functions: *no...except Leo and Lea* denotes an atom of the algebra of intersective function determined by the set composed of two elements, Leo and Lea.

The algebra INT contains a sub-algebra $CARD$ of cardinal determiners: they are denotations of, roughly speaking, various numerals. By definition:

D4: $F \in CARD$ iff for all properties X, Y, W and Z, if $|X \cap Y| = |W \cap Z|$ then $F(X)(Y) = F(W)(Z)$.

Atoms of $CARD$ are the functions f_α, where α is a cardinal, such that $f_\alpha(X)(Y)$ is true iff $|X \cap Y| = \alpha$. From this definition it follows that any cardinal determines an atom of $CARD$. Thus the determiner *exactly n* denotes an atomic cardinal function.

As might be expected the algebra $CO\text{-}INT$ has an analogous sub-algebra. This is the algebra $CO\text{-}CARD$ of co-cardinal functions (Keenan 1993):

D5: $F \in CO\text{-}CARD$ iff for all properties X, Y, W and Z, if $|X - Y| = |W - Z|$ then $F(X)(Y) = F(W)(Z)$

Determiners like *every...except five* denote co-cardinal functions. Moreover, every post-complement of a cardinal quantifier is a co-cardinal quantifier.

Unary determiners whose semantics also involves cardinality of sets denoted by arguments are those denoting *proportional quantifiers* $PROPORT$. They are defined as follows (Keenan 2002):

D6: $F \in PROPORT$ iff for all sets $X, Y, W, Z \subseteq E$ if $|W| \times |X \cap Y| = |X| \times |W \cap Z|$ then $F(X)(Y) = F(W)(Z)$

Proportional quantifiers form an atomic Boolean algebra (Zuber 2005).

A classical example of a proportional quantifier is the quantifier denoted by the determiner *most* (in the sense of *more than half*).

Let me mention in addition the algebra of generalised cardinals or $GCARD$ introduced in Keenan and Faltz 1975 (under the name of cardinal independent) and, independently in Zuber 2004 and studied in more detail in Zuber 2005. This algebra, and its higher order generalisations will play an essential role in what follows. By definition:

D7 $F \in GCARD$ iff for all properties X, Y, Z if $|X \cap Y| = |X \cap Z|$ then $F(X)(Y) = F(X)(Z)$.

What definition D7 says intuitively is that a generalised cardinal is a function which cannot distinguish among properties Y_1 and Y_2 at the argument X if $X \cap Y_1$ and $X \cap Y_2$ have the same cardinality.

Obviously the algebra $GCARD$ is a proper sub-algebra of $CONS$ and contains as proper sub-algebras $CARD$ and, only in finite universes, $CO\text{-}CARD$. Moreover the following fact is also true (Zuber 2005):

Fact 7: $PROPORT$ is a sub-algebra of $GCARD$.

Proof: It is enough to show that any proportional quantifier is a generalised cardinal. Suppose that D is proportional and that for arbitrary X, Y and Z one has $|X \cap Y| = |X \cap Z|$. Then it is also true that $|X| \times |X \cap Y| = |X| \times |X \cap Z|$. Since D is proportional this means that $F(X)(Y) = F(X)(Z)$ and thus D is generalised cardinal.

Finally the algebra of $GCARD$ has a sub-algebra of those quantifiers which are at the same time fixed points with respect to post-complements ($FPPCPL$) that is such type $\langle 1, 1 \rangle$ quantifiers D that $D(X)(Y) = D(X)(Y')$ (*half* is a typical determined which denotes such quantifiers). They can be defined as follows (Zuber 2005):

D8: $D \in GCARD \cap FPPCPL$ iff $D(X)(Y_1) = D(X)(Y_2)$ whenever $|X \cap Y_1| = |X \cap Y_2'|$.

I mentioned the algebra $GCARD$ because, as indicated by above properties, many classes of conservative quantifiers are generalised cardinals. Let us see some examples. Quantifiers NO and $FIVE$ are generalised cardinals because they are cardinals. Similarly $EVERY$ and $EVERY...EXCEPT$ 10 are generalised cardinals because they are co-cardinals. Notice that this last clain is true only for finite universes since only in this case it is true that $|X \cap Y_1| = |X \cap Y_2|$ iff $|X \cap Y_1'| = |X \cap Y_2'|$.

Consider now the determiner *the n*. One considers traditionally that *the n* denotes the quantifier $THE\ n$ defined as follows: $THE\ n(X)(Y) = 1$ iff $|X| = n$ and $X \subseteq Y$. Observe that $THE\ n$ is neither cardinal nor co-cardinal. One can check also that it is not proportional: to see this (for $n = 1$) take X, Y, W, Z such that $|X| = 1$, $|W| = 2$, $X \subseteq Y$ and $W \subseteq Z$. It is easy to see, however, that *the n* denotes a proper generalised cardinal quantifier. Similarly one can show that the determiners like *most but less than 10* or *most or at least 10* are properly generalised cardinals.

Let us show now that the quantifier $MOST\ OR\ AT-LEAST(10)$ is neither cardinal nor proportional. We show first that it is not cardinal. Suppose for this that $|X \cap Y| = |W \cap Z| < 10$, $|X \cap Y| \leq |X \cap Y'|$ and $|W \cap Z| > |W \cap Z'|$. In this case $MOST(X)\ OR\ AT-LEAST(10)(X)(Y) = 0$ and $MOST(W)\ OR\ AT-LEAST(10)(W)(Z) = 1$ which means that $MOST\ OR\ AT-LEAST(10)$ is not cardinal.

Suppose now that $|X \cap Y| = |X \cap Y'| = 10$, $|W \cap Z| = 8$, $|X| = 20$ and $|W| = 16$. In this case $|W \times |X \cap Y| = |X| \times |W \cap Z|$. However in this case $MOST(X)\ OR\ AT-LEAST(10)(X)(Y) = 1$ and $MOST(W)\ OR\ AT-LEAST(10)(W)(Z) = 0$ which means that the considered quantifier is not proportional. It is, however, generalised cardinal because it is a join of two generalised cardinals.

Observe finally that *(exactly) half ... out of n* is a generalised cardinal which is a member of $FPPCPL$ (and which is neither proportional nor cardinal).

Important point in the context of unary determiners and above examples is that not all of them denote generalised cardinals. To see this it is enough to take a properly intersective (non-cardinal) or a properly co-intersective (non-co-cardinal) quantifiers. For instance *No...except Leo, most/some ...including Leo* and *every... except Leo* are not generalised cardinals. My point is to try to show, however, that all binary determiners denote (appropriately generalised to the higher order case) generalised cardinals.

We can now introduce two other classes of type $\langle 1, 1 \rangle$ quantifiers: *symmetric* and *contrapositional* ones. When extended to higher order quantifiers they will play essential role in our analysis or binary determiners. In addition they allow us to better understand the relationship between conservative quantifiers in general and various their sub-classes.

Symmetric and contrapositional type $\langle 1, 1 \rangle$ quantifiers are defined as follows (Zuber 2007):

D9: $F \in SYM$ iff for all properties X, Y one has $F(X)(Y) = F(Y)(X)$
D10: $F \in CONTR$ iff for all sets X, Y one has $F(X)(Y) = F(Y')(X')$.

Both sets, SYM and $CONTR$ form atomic Boolean algebras. They elements need not to be conservative. The following propositions show when they are conservative (Zuber 2007):

Proposition 8: $CONS \cap SYM = INT$
Proposition 9: $CONS \cap CONTR = CO\text{-}INT$

It follows from Propositions 8 and 9 that under conservativity type $\langle 1, 1\rangle$ symmetric quantifiers are the same as the intersective ones and contrapositional are the same as co-intersective ones. As we will see this is not the case for higher order quantifiers.

Interestingly we have similar relations between algebras $GCARD$, $CARD$, $CO\text{-}CARD$ and SYM and $CONTR$. More precisely, the following propositions hold (Zuber 2007):

Proposition 10: $GCARD \cap SYM = CARD$
Proposition 11: $GCARD \cap CONTR = CO\text{-}CARD$

Thus cardinal and intersective quantifiers are symmetric. For example $FIVE$, $SOME$, $SOME..., INCLUDING\ LEA$ and $NO, ..., EXCEPT\ LEO$ are symmetric quantifiers.

It is interesting that one can define symmetric quantifiers in the format we use in other definitions and which can be easily generalised to quantifiers of higher types. The following trivial proposition which indicates such an equivalent definition will be used as a convenient handle for generalising symmetry to higher types :

Proposition 12: $F \in SYM$ iff there exists a commutative binary function "$\otimes$" taking sets as arguments such that for all X, Y, W, Z if $X \otimes Y = W \otimes Z$ then $F(X)(Y) = F(W)(Z)$.

Similar property holds for contrapositional quantifiers:

Proposition 13: $F \in CONTR$ iff there exists a commutative binary function "$\otimes$" taking sets as arguments such that for all X, Y, W, Z if $X \otimes Y' = W \otimes Z'$ then $F(X)(Y) = F(W)(Z)$.

Propositions 12 and 13 will allow us to generalise the notion of symmetry and contraposition to quantifiers of higher types.

3 Quantifiers of higher types

In the previous section we presented various properties quantifiers of type $\langle 1, 1\rangle$ quantifiers that is quantifiers corresponding to binary relations between sets. They are denotations of unary determiners. Since we are going to make some claims about constraints on denotations of binary determiners we need to consider how to extend various properties discussed in the previous section to a more general case of quantifiers which are still binary relations but arguments of these relations are other relations, usually binary, or sets and binary relations. Though we are basically interested in denotations of binary determiners and thus in binary relations having between binary relations and sets in most definitions we will propose we will not limit the number of arguments corresponding to nominal arguments of determiner. Thus we define various properties of higher order quantifiers in the way that they are applicable to denotations of n-ary determiners in general, for n

arbitrary. This move will allow us to understand better the basic ideas underlying various definitions. Thus most definitions to be given concern type $\langle 1^n, 1\rangle$ quantifiers that is binary relations whose first argument is an n-ary relation between sets and the second argument is a set.

Most of the definitions we are looking for have already been suggested (cf. Keenan and Moss 1984, Beghelli 1994, Keenan 2003, Zuber 2005). After the discussion from the previous section we have a relatively clear intuition of how to define conservative quantifiers of higher types. Here are two general definitions: the definition of conservative quantifiers and the definition of straightforwardly related generalised cardinals (Zuber 2005):

D11: $D \in CONS_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_1, Y_2$, $D(X_1, ..., , X_n)(Y_1) = D(X_1, ..., X_n)(Y_2)$ if $X_i \cap Y_1 = X_i \cap Y_2$, for every $1 \leq i \leq n$.
D12: $D \in GCARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_1, Y_2$, $D(X_1, ..., X_n)(Y_1) = D(X_1, ..., X_n)(Y_2)$ if $|X_i \cap Y_1| = |X_i \cap Y_2|$, for every $1 \leq i \leq n$.

As in the case of simple quantifiers conservative higher order quantifiers can be defined in two other ways. This is indicated by the following facts;

Fact 14: $D \in CONS_{\langle 1^n,1\rangle}$ iff $D(X_1, ..., X_2)(Y) = D(X_1, ..., X_n)(Y \cap \bigcup_n X_i)$, for every $1 \leq i \leq n$.
Fact 15: $D \in CONS_{\langle 1^n,1\rangle}$ iff $D(X_1, ..., X_2)(Y) = D(X_1, ..., X_n)(Y \cup \bigcap_n X_i')$, for every $1 \leq i \leq n$.

These facts follow from the equalities given in (3):

(3) $X_i \cap Y = X_i \cap Y \cap \bigcup_n X_i = X_i \cap (\bigcap_n X_i' \cup Y)$

It is not difficult to establish that:

Fact 16: $GCARD_{\langle 1^n,1\rangle} \subset CONS_{\langle 1^n,1\rangle}$

Both sets, $GCARD_{\langle 1^n,1\rangle}$ and $CONS_{\langle 1^n,1\rangle}$, form atomic Boolean algebras. More specifically we have (Zuber 2005):

Proposition 17: Let $1 \leq i \leq n$, $P_i \subseteq E$ and $P \subseteq \bigcup_i P_i$. Then the function $F_{P_1,...,P_n,P}$ such that $F_{P_1,...,P_n,P}(X_1, ..., X_n)(Y) = 1$ iff $X_i = P_i$ and $P = Y \cap \bigcup_i X_i$ is an atom of $CONS_{\langle 1^n,1\rangle}$. All atoms of $CONS_{\langle 1^n,1\rangle}$ are of this form.

Concerning atoms of $GCARD_{\langle 1^n,1\rangle}$ we have:

Proposition 18: Let $1 \leq i \leq n$, $P_i \subseteq E$. Then the function $F_{P_1,...,P_n,n}$ such that $F_{P_1,...,P_n,n}(X_1, ..., X_n)(Y) = 1$ iff $P_i = Xi$ and $|X_i \cap Y| = n$ is an atom of $GCARD_{\langle 1^n,1\rangle}$. All atoms of $GCARD_{\langle 1^n,1\rangle}$ are of this form.

Proof: We prove only the first part of the proposition. Suppose *a contrario* that the function $F_{P_1,...,P_n,n}$ is not an atom of $GCARD_{\langle 1^n,1\rangle}$. This means that there exist a function G of type $\langle 1^n, 1\rangle$ which is a generalised cardinal and which is strictly included in $F_{P_1,...,P_n,n}$. This entails that for some $X_1..., X_n, Y$ we have

$G(X_1, ..., X_n)(Y) = 0)$ and $F_{P_1,...,P_n,n}(X_i, ..., X_n, Y) = 1$. Consequently, given the definition of $F_{P_1,...,P_n,n}$ we have $P_i = X_i$ and $|X_i \cap Y| = n$. Since G cannot be a constant function (always equal 0) there is a sequence of arguments $W_1, ..., W_n, Z$ such that $G(W_1, ..., W_n), (Z) = 1$ and $F_{P_1,...,P_n,n}(W_1, ..., W_n), (Z) = 1$. It follows from this that $P_i = W_i$ and $|W_i \cap Z| = n$. But this is impossible because in this case G would not be a generalised cardinal.

Proposition 18 allows us to calculate the number of generalised cardinals of type $\langle 1^n, 1\rangle$. Thus we have:

Proposition 19: Let $|E| = k$. Then the number of atoms of $GCARD_{\langle 1^n,1\rangle}$ (over E)= $\sum_{i=0}^{k}[(k-i+1) \times \left(\begin{array}{c} k \\ i \end{array} \right)]^n$

Proposition 19 allows us to establish the number of elements of $GCARD_{\langle 1^n,1\rangle}$. Thus if $|E| = 2$ and $n = 2$, the algebra $GCARD_{\langle 1^n,1\rangle}$ has 26 atoms and thus 2^{26} elements. Keenan and Moss (1985) show that in this case the algebra $CONS_{\langle 1^2,1\rangle}$ contains 2^{47} elements.

Comparing the above definitions with the definitions of intersective, co-intersective, cardinal and co-cardinal simple quantifiers we see how to define higher order intersective, co-intersecive, cardinal and co-cardinal quantifiers of type $\langle 1^n, 1\rangle$. Thus we have the following definitions:

D 13: $D \in INT_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $X_i \cap Z_1 = Y_i \cap Z_2$ then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$, for every $1 \leq i \leq n$.
D 14: $D \in CO\text{-}INT_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $X_i - Z_1 = Y_i - Z_2$, for every $1 \leq i \leq n$ then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$
D15: $D \in CARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $|X_i \cap Z_1| = |Y_i \cap Z_2|$, for every $1 \leq i \leq n$ then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$
D16: $D \in CO\text{-}CARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2, D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$ if $|X_i - Z_1| = |Y_i - Z_2|$, for every $1 \leq i \leq n$.

Various classes of quantifiers specified in these definitions are related between themselves. We have in particular (cf. Keenan and Moss 1984):

Fact 20: $CONS_{\langle 1^n,1\rangle}$, $GCARD_{\langle 1^n,1\rangle}$, $INT_{\langle 1^n,1\rangle}$, $CO\text{-}INT_{\langle 1^n,1\rangle}$, $CARD_{\langle 1^n,1\rangle}$, $CO\text{-}CARD_{\langle 1^n,1\rangle}$ form Boolean algebras.
Fact 21: $CARD_{\langle 1^n,1\rangle} \subseteq INT_{\langle 1^n,1\rangle} \subseteq CONS_{\langle 1^n,1\rangle}$

Another relation easy to establish (holding in finite models), and analogous to that established in the previous section of quantifiers of type $\langle 1, 1\rangle$ is indicated in:

Fact 22: $CARD_{\langle 1^n,1\rangle} \cup CO\text{-}CARD_{\langle 1^n,1\rangle} \subseteq\ GCARD_{\langle 1^n,1\rangle} \subseteq CONS1_{\langle 1^n,1\rangle}$

To be more precise the above set inclusions can in fact be replaced by statements indicating that included sets are sub-algebras of sets in which they are included.

It is easy to check that the quantifier $MORE...THAN...$ denoted by the binary determiner *more ...than...* (occurring in NPs on the subject position) is a type

$\langle 1^2, 1\rangle$ cardinal quantifier. We will discuss many other examples in the next section.

All the above definitions specify various classes of type $\langle 1^n, 1\rangle$ quantifiers. Natural languages have also clear cases of type $\langle 1, 1^n\rangle$ quantifiers or at least of type $\langle 1, 1^2\rangle$. An example of such a quantifier is the quantifier $MORE...ARE...THAN...$ (as the denotation of the determiner occurring in *More students are Buddhists than shogi players*. This means that various classes of type $\langle 1, 1^n\rangle$ quantifiers should be defined as well. We give here only definitions of conservativity and intersectivity for such quantifiers. These definitions, in conjunction with other definitions given above show how to define other properties of type $\langle 1, 1^n\rangle$ quantifiers (cf. Zuber 2005):

D17: $D \in CONS_{\langle 1,1^n\rangle}$ iff for all X, Y_i, Z_i if $X \cap Y_i = X \cap Z_i$, for every $1 \leq i \leq n$ then $D(X)(Y_1, ..., Y_n) = D(X)(Z_1, ..., Z_n)$
D18: $D \in INT_{\langle 1,1^n\rangle}$ iff for all X_1, X_2, Y_i, Z_i, $D(X_1)(Y_1, ..., Y_n) = D(X_2)(Z_1, ..., Z_n)$, whenever $X_1 \cap Y_i = X_2 \cap Z_i$, for every $1 \leq i \leq n$.

Higher type quantifiers, in addition to their Boolean complements, have also post-complements. A general definition of the post-complement of type $\langle 1^k, 1^l\rangle$ quantifier goes as follows:

D 19: If F is a type $\langle 1^k, 1^l\rangle$ quantifier then $F\neg$, the post-complement of F, is that type $\langle 1^k, 1^l\rangle$ quantifier for which $F\neg(X_1, ..., X_k)(Y_1, ..., Y_l) = F(X_1, ..., X_k)(Y_1', ..., Y_l')$

The operation of post-complementation has similar effects in the case of higher type quantifiers, as in the case of simple quantifiers. For instance:

Fact 23: $D \in INT_{\langle 1^n,1\rangle}$ iff $D\neg \in CO\text{-}INT_{\langle 1^n,1\rangle}$ and $D \in INT_{\langle 1,1^n\rangle}$ iff $D\neg \in CO\text{-}INT_{\langle 1,1^n\rangle}$

Similar conditions hold for cardinal and co-cardinal quantifiers.

We want now to define symmetry and contraposition for quantifiers of higher order types. Obviously in such definitions we cannot just permute arguments of the corresponding relation since such a permutation changes the type of quantifier in this case. For indeed, if the type of a relation depends on the type of its arguments then the binary relation which has as a first argument a set and as the second argument an n-ary relation between sets is not of the same type as the binary relation whose first argument is a n-ary relation between sets and the second argument is a set. But this just means that we cannot define symmetric higher order quantifiers by comparing relations with permuted arguments as in definition D9 for simple quantifiers. It is possible, however, in this case to use the equivalence indicated in Proposition 12 and 13 and define symmetric and contrapositional quantifiers of higher types in the definitional format mostly used here (Zuber 2007):

D 20: A type $\langle 1^n, 1\rangle$ quantifier D is symmetric iff there exists a binary commutative function $\otimes$ on pairs of sets such that $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$ if $X_i \otimes Z_1 = Y_i \otimes Z_2$, for every $1 \leq i \leq n$.
D 21: A type $\langle 1^n, 1\rangle$ quantifier D is contrapositional iff there exists a binary

commutative function $\otimes$ such that $\forall X_i, Y_i, Z_1, Z_2$, if $X_i \otimes Z_1' = Y_i \otimes Z_2'$ then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$, for every $1 \leq i \leq n$.

Similarly for type $\langle 1, 1^n \rangle$ quantifiers:

D 22: A type $\langle 1, 1^n \rangle$ quantifier is symmetric iff there exists a binary commutative function $\otimes$ on pairs of sets such that for all X_1, X_2, Y_i, Z_i, if $X_1 \otimes Y_i = X_2 \otimes Z_i$, then $D(X_1)(Y_1, ..., Y_n) = D(X_2)(Z_1, ..., Z_n)$, for every $1 \leq i \leq n$.
D 23 A type $\langle 1, 1^n \rangle$ quantifier is contrapositional iff there exists a binary commutative function $\otimes$ on pairs of sets such that for all X_1, X_2, Y_i, Z_i, $D(X_1)(Y_1, ..., Y_n) = D(X_2)(Z_1, ..., Z_n)$, whenever $X_1 \otimes Y_i' = X_2 \cap Z_i'$, for every $1 \leq i \leq n$.

The following propositions show the usefulness of these definitions for n-ary determiners denoting symmetric or contrapositional quantifiers (Zuber 2007):

Proposition 24: Let $F \in PDET_{\langle 1^n, 1 \rangle}$ and $G \in PDET_{\langle 1, 1^n \rangle}$ such that $F(X_1, ..., X_n)(Y) = G(Y)(X_1, ..., X_n)$. Then F is symmetric iff G is symmetric.
Proposition 25: Let $F \in PDET_{\langle 1^n, 1 \rangle}$ and $G \in PDET_{\langle 1, 1^n \rangle}$ such that $F(X_1, ..., X_n)(Y) = G(Y')(X_1', ..., X_n')$. Then F is contrapositional iff G is contrapositional.

Thus, roughly, proposition 24 says that if two functions have "symmetric types" and are equal then they are both symmetric.

Concerning symmetric quantifiers the following easy to establish fact (formulated somewhat informally) allows us to understand what symmetry for higher types of quantifiers means:
Fact 26: A type $\langle 1^n, 1 \rangle$ quantifier D is symmetric iff $D(X, ...X)(Y) = D(Y, ..., Y)(X)$

We observe also, as for simple quantifiers, that symmetric and contrapositional quantifiers are related by post-complements:

Fact 27: A higher type quantifier F is symmetric iff $F\neg$ is contrapositional.

Furthermore we can also easily establish the following:

Fact 28: If $F \in INT_{\langle 1^n, 1 \rangle}$ or $F \in INT_{\langle 1, 1^n \rangle}$ then F is symmetric.
Fact 29: if $F \in CO\text{-}INT_{\langle 1^n, 1 \rangle}$ or $F \in CO\text{-}INT_{\langle 1, 1^n \rangle}$ then F is contrapositional.

We can illustrate proposition 19, and indirectly fact 22 by the following example. We will consider two quantifiers we have already seen. Consider first the type $\langle \langle 1, 1 \rangle, 1 \rangle$ quantifier $MORE...THAN...$. It is denoted by the binary determiner *more...than...* as it occurs in *More students than teachers*. Its semantics is given by: $MORE(X_1, X_2)(Y) = 1$ iff $|X_1 \cap Y| > |X_2 \cap Y|$. As the second quantifier consider $MORE...ARE...THAN...$. This type $\langle 1, \langle 1, 1 \rangle \rangle$ quantifier is the denotation of the complex (structured) determiner found in *More students are vegetarians than (are) Buddhists*. Its semantics is given by $MORE(Y)(X_1, X_2) = 1$ iff $|Y \cap X_1| > |Y \cap X_2|$. It follows from the provided semantics that both quantifiers are intersective (in fact *cardinal*) and thus symmetric. Furthermore, we can check that they have both identical semantics up to the permutation of arguments required by the symmetry. So in particular *More students than teachers* *are Buddhists* is equivalent to *More Buddhists are students than teachers.*

To illustrate proposition 10 we can use the example of higher type (reducible) quantifiers formed by the quantifier $EVERY$ and the conjunction AND. Thus *Every student and teacher is a Buddhist* is equivalent, under the distributive reading, to *Every non Buddhist is a non-student and a non-teacher.* This means that the type $\langle\langle 1,1\rangle,1\rangle$ quantifier $EVERY...AND...IS...$ and the type $\langle 1,\langle 1,1\rangle\rangle$ quantifier $EVERY...IS...AND...$ are both contrapositional.

The remain the definition of proportional quantifiers to be given. I give it only for the case when $n=2$. The definition goes as follows (Zuber 2005):

D24: $D \in PROPORT_{\langle 1^2,1\rangle}$ iff for all $X_1, X_2, Y_1, Y_2, Z_1, Z_2$, $D(X_1,X_2)(Z_1) = D(Y_1,Y_2)(Z_2)$ whenever $|Y_1|\times|Y_2|\times|X_1\cap Z_1| = |X_1|\times|X_2|\times|Y_1\cap Z_2|$ and $|Y_1|\times|Y_2|\times|X_2\cap Z_1| = |X_1|\times|X_2|\times|Y_2\cap Z_2|$.

One checks by calculation that according to D 24 determiners like *proportionally as many... as...* denote proportional type $\langle 1^2,1\rangle$ quantifiers.

For proportional quantifiers the following is true:

Proposition 30: $PROPORT_{\langle 1^2,1\rangle}$ is a sub-algebra of $GCARD_{\langle 1^2,1\rangle}$.

To conclude this section let me mention some differences between simple and higher order quantifiers. There are many such differences. One important difference concerns simple et binary proportional quantifiers: only the former are closed with respect to post-complement. Other differences, more important for our analysis, concern intersective and symmetric quantifiers. We have seen that under conservativity these to notions are co-extensive. Furthermore we have also seen that intersectivity if simple quantifiers can be defined in four equivalent ways (cf. Fact 5). This is not the case for intersectivity of type $\langle\langle 1,1\rangle 1\rangle$ or type $\langle 1\langle 1,1\rangle\rangle$ quantifiers. In particular there are symmetric and conservative type $\langle\langle 1,1\rangle 1\rangle$ quantifieres which are not intersective. Here is a simple example. Let D be a type $\langle\langle 1,1\rangle 1\rangle$ quantifier defined as follows $D(X_1,X_2)(Y)=1$ iff $X_1\cup Y = X_2\cup Y$. It is easy to show that this quantifier is symmetric and conservative but not intersective. Its symmetry is shown by choosing as the commutative operation $\otimes$ the operations of set-union $\cup$.

4 Denotations of binary determiners

Binary determiners are discontinuous functional expressions which take two arguments. They are not necessarily "nominal" because syntactically they can be of two categories. First, they can take two common nouns and form a noun phrase. Though such NPs can occur on various positions we will consider only the case when they occur on subject position. Thus determiners of the first category form a sentence with two com moun nouns and a verb phrase. Consequently, semantically, they denote type $\langle\langle 1,1\rangle,1\rangle$ quantifiers. Second, binary determiners can take two verb phrases and form with one common noun a sentence. In this case they denote $\langle 1,\langle 1,1\rangle\rangle$ quantifiers. Our proposal here concerns basically binary determiners of the first category.

From the formal and empirical view it is useful to distinguish two types of binary determiners (cf. Keenan and Moss 1985): comparative binary determiners and Booleanly reducible binary determiners. Often discussed in the literature "natural" higher order quantifiers are so-called *comparative binary* quantifiers (Beghelli 1992). These are quantifiers denoted in the simplest case by discontinuous (binary) determiners like *more... than* or *as many... as*. When these determiners form subject NPs (in sentences with "simple" VPs) then they denote quantifiers of type $\langle\langle 1,1\rangle 1\rangle$. These quantifiers can be said to be genuine higher order since they cannot be reduced to a Boolean combination of simple quantifiers (Keenan and Moss 1985, Beghelli 1994). Furthermore. they are "natural" in the sense that the determiners by which they are denoted have a categorial status of binary determiners syntactically justified (Keenan 1989).

Beghelli (1994) distinguishes various sub-groups of comparative determiners (quantifiers). Usually they exhibit a complex syntax which can be ignored for our purposes. The simplest and in some sense basic group of determiners may be called *simple comparatives*. They include determiners like *more...than..., exactly as many... as..., the same number of..as..,* , etc. It is easy to see that these determiners denote cardinal quantifiers and thus, given proposition x, they are symmetric and, at the same time, generalised cardinals. Let us show this for illustration on one example. Consider for instance the quantifier $FEWER...THAN...$ denoted by the determiner occurring in the noun phrase *fewer students than teachers*. Its semantics is given in (1):

(1) $FEWER(X_1)THAN(X_2)(Y) = 1$ iff $|X_1 \cap Y| < |X_2 \cap Y|$

To show that it is cardinal suppose that (2) and (3) hold. We have to show that (4) holds as well:

(2) $|X_1 \cap Y_1| = |X_1 \cap Y_2|$ and $|X_2 \cap Y_1| = |X_2 \cap Y_2|$
(3) $FEWER(X_1)THAN(X_2)(Y_1) = 1$
(4) $FEWER(X_1)THAN(X_2)(Y_2) = 1$

The result is obvious given the semantics in (1): the equalities in (2) allow us to make replacements making (4) true.

Since cardinal quantifiers (of any type) form a Boolean algebra they are closed with respect to Boolean operations. This means for instance that $AT-LEAST-AS-MANY...AS...$ is also cardinal because it is the complement of $FEWER...THAN...$. Similarly the complex determiner *at least 10 more but not more than 20 more... than ...* (as it occurs in *at least ten more but not more then 20 students more then teachers*) denotes the meet of two quantifiers denoted respectively by *at least 10 more... than...* and *not more than 20 ...than...*. Since both these quantifiers are cardinal the whole quantifier is also a cardinal and consequently symmetric and generalised cardinal. Symmetry of many other quantifiers can be established in the same way (see Beghelli 1994).

One observes in addition that for many simple comparative determiners mentioned above there exist logically equivalent syntactically more complex ones. For instance *a lesser number of... than...* is semantically equivalent to *fewer...than...* and *exactly as many... as...* is equivalent to *exactly the same number of... as...*. Their semantic status is however the same.

There is a class of binary determiners which denote intersective but not cardinal quantifiers. These are determiners which may be called *modified comparatives*.

They can be modified by adjectives (as in *more male... than female...* or by possessives (as in *more Leo's... than Bill's...*). What is interesting is the fact that though such modified comparatives are in some sense derived from cardinal ones they are not cardinal. In other words the modification does not preserve the property of being cardinal. Modification preserves, however, intersectivity. Let us se this in more details.

Let us observe first that modifiers we are talking about denote absolute functions (absolute modifiers). A function M from sets to sets is *absolute* (Keenan and Faltz 1985) iff for any set X, $F(X) = X \cap M(E)$. Absolute adjectives (*male, female*) and possessives denote absolute modifiers. Thus, roughly *female students* are *students and female objects* and *Bill's bicycles* are *bicycles and Bill's objects*.

Let us define now an intersective type $\langle\langle 1,1\rangle,1\rangle$ quantifier restricted (mopdified) by two sets:

D25: Let A and B be sets and D a type $\langle\langle 1,1\rangle,1\rangle$ quantifier. Then $D_{A,B}$ is a type $\langle\langle 1,1\rangle,1\rangle$ quantifier defined as follows: $D_{A,B}(X_1.X_2)(Y) = D(A\cap X_1, B\cap X_2)(Y)$

For such modified quantifiers it is easy to establish the following fact:

Fact 30: If $D \in INT_{\langle 1^2,1\rangle}$ then $D_{A,B} \in INT_{\langle 1^2,1\rangle}$, for any set A, B.

It follows from fact 30 that modified comparative binary determiners denote intersective quantifiers (because they are obtained by modification of cardinal, and thus intersective quantifiers). Consequently modified comparative binary determiners also denote symmetric (but not generalised cardinal) quantifiers.

Beghelli (1994) mentions also existence of the class of binary determiners he calls *identity comparative*. These determiners do not involve comparison of cardinalities or quantities but rather a comparison of identities of individuals. syntactically they combine one common noun with two VPs to form a sentence and thus they denote type $\langle 1,\langle 1,1\rangle\rangle$ quantifiers. Here are some (Beghelli's) examples of sentences with such determiners:

(5) The same students came early as left late.
(6) Whatever students came early left late.
(7) The same five students came early as left late.

The determiners in the above sentences denotes the following quantifiers:

(8) $THE\text{-}SAME(X)(Y_1, Y_2) = 1$ iff $X \cap Y_1 = X \cap Y_2$
(9) $WHATEVER(X)(Y_1, Y_2) = 1$ iff $X \cap Y_1 \subseteq X \cap Y_2$
(10) $THE\text{-}SAME\text{-}5(X)(Y_1, Y_2) = 1$ iff $X \cap Y_1 = X \cap Y_2 \wedge |X \cap Y_1| = 5$

Thus sentence (6) is true iff the set of students who came early is included in the set of students who left late. It is easy to show that quantifiers in (8), (9) and (10) are all intersective and thus symmetric.

There remains a last type of determiners we need to examine is represented by the proportional binary determiners as the one found in (11). It denotes a type $\langle\langle 1,1\rangle,1\rangle$ quantifier which has the semantics given in (12):

(11) Proportionally as many students as teachers danced.
(12) $PROP\text{-}AS\text{-}MANY(X_1, X_2)(Y) = 1$ iff $|X_1 \cap Y|/|X_1| = |X_2 \cap Y|/|X_2|$

The quantifier in (12) is proportional in the sense of D24. It is not intersective. We show, not quite explicitly, that it is symmetric. According to D22 we have to show that there exists a binary operation on sets $\otimes$ which is commutative and such that if (13) holds then (14) holds:

(13) $X_1 \otimes Z_1 = Y_1 \otimes Z_2$ and $X_2 \otimes Z_1 = Y_2 \otimes Z_2$
(14) $PROP\text{-}AS\text{-}MANY(X_1, X_2)(Z_1) = PROP\text{-}AS\text{-}MANY(Y_1, Y_2)(Z_2)$

Chose $\otimes$ as: $X \otimes Y = |X \cap Y|/(|X| \times |Y|)$. This operation is obviously commutative. A somewhat tedious simple arithmetic operations on fractions in conjunction with necessary substitutions of equals by equals lead to the required equivalence in (14).

Examples:
Apart fom Leo all students who are Buddhists are vegetarians

5 Conclusion

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