Toward a Logic of Cumulative Quantification

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Abstract

This paper studies entailments between sentences like *three boys kissed five girls* involving two or more numerically quantified noun phrases that are interpreted as expressing *cumulative quantification* in the sense of Scha (1984). A precise characterization of when one such sentence entails another such sentence that differs from it only in the numerals is crucially important to evaluate claims about *scalar implicatures* arising from the use of those sentences, as pointed out by Shimada (to appear). This problem turns out to be non-trivial and surprisingly difficult. We give a characterization of these entailments for the case of sentences with two noun phrases, together with a complete axiomatization consisting of two simple inference rules. We also give some valid inference rules for sentences with three noun phrases.

1 Introduction

This paper concerns sentences expressing *cumulative quantification* (Scha, 1984), exemplified by (1):

$$\text{Three boys kissed five girls.} \tag{1}$$

We call sentences like (1) *cumulative sentences*, focusing only on their cumulative reading. In general, a cumulative sentence may involve k numerically quantified noun phrases and a verb expressing a k-ary relation. A general form of a cumulative sentence may be schematically represented by

$$n_1 \ \mathrm{N}_1 \ \mathrm{V} \ n_2 \ \mathrm{N}_2 \ \dots n_k \ \mathrm{N}_k, \tag{2}$$

where each n_i is a number word, N_i is a count noun, and V is a verb.

Following Krifka (1999), we assume that the relevant truth conditions of sentences of the form (2) are as given in (3), where we write $\pi_i(R)$ for the ith projection $\{\, x_i \mid (x_1, \dots, x_k) \in R \,\}$ of a k-ary relation R:

$$\exists X_1 \dots \exists X_k \bigwedge_{i=1}^{k} \Big(|X_i| = n_i \wedge X_i \subseteq [\![\mathrm{N}_i]\!] \wedge X_i = \pi_i([\![\mathrm{V}]\!] \cap (X_1 \times \cdots \times X_k)) \Big). \tag{3}$$

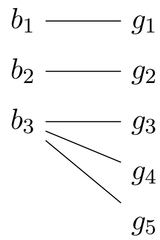

Figure 1: A model where *three boys kissed five girls* and *three boys kissed four girls* are true but *two boys kissed five girls* is false.

The truth conditions in (3) are deliberately weaker than those suggested by Scha (1984), which may be represented as follows:

$$\bigwedge_{i=1}^{k} |\pi_i([\![\mathrm{V}]\!] \cap ([\![\mathrm{N}_1]\!] \times \cdots \times [\![\mathrm{N}_k]\!]))| = n_i. \tag{4}$$

Krifka's (1999) idea was that (3) represents the truth conditions that are directly delivered by compositional semantics involving plural predication,[1] but they are pragmatically strengthened by scalar implicatures, resulting in something like (4).

In more detail, Krifka (1999) assumes that when a speaker utters a sentence of the form (2) for a specific choice of $\mathrm{N}_1, \ldots, \mathrm{N}_k, \mathrm{V}$, all the sentences of the form (2) for the same choice of $\mathrm{N}_1, \ldots, \mathrm{N}_k, \mathrm{V}$ constitute the set of relevant alternatives for the sentence, partially ordered by the entailment relation. According to Krifka, Grice's maxim of Quantity has the effect that the numbers $n_1, \ldots, n_k$ actually used in the utterance must be the highest numbers that make the sentence true.

A problem with this account, as pointed out by Shimada (to appear), is that the partial order on $\mathbb{N}^k$ given by the entailment relation between the truth conditions (3) is not the familiar coordinatewise order defined by

$$(n_1, \ldots, n_k) \leq (m_1, \ldots, m_k) \Longleftrightarrow n_1 \leq m_1 \wedge \cdots \wedge n_k \leq m_k.$$

For instance, *three boys kissed five girls*, understood according to (3), entails *three boys kissed four girls*, but not *two boys kissed five girls* (see Figure 1). This means that a speaker who knows the first of these three sentences to be true may nevertheless choose to utter the last sentence to convey information not carried by the former. There is no reason, then, to expect that Grice's maxim forces the speaker to choose the highest

[1] Representing plural individuals as sets of atomic individuals, we may suppose that if the bare form of a verb V denotes a k-ary relation $[\![\mathrm{V}]\!]$ on atomic individuals, its plural form denotes its closure under *cumulativity*, which amounts to $\{\, (X_1, \ldots, X_k) \mid \bigwedge_{i=1}^{k} X_i = \pi_i([\![\mathrm{V}]\!] \cap (X_1 \times \cdots \times X_k)) \,\}$. Numerically quantified noun phrases $n_i\ \mathrm{N}_i$ would denote generalized quantifiers $\{\, P \mid \exists X (X \subseteq [\![\mathrm{N}_i]\!] \wedge |X| = n \wedge X \in P) \,\}$ over plural individuals and combine with the plural verb denotation in the standard way to produce (3).

numbers that make the sentence true. Shimada (to appear) claims that this is as it should be, and an utterance of *two boys kissed five girls* in fact does not implicate the negation of *three boys kissed five girls*.

The actual entailment relation between cumulative sentences understood according to the truth conditions in (3), even for the case $k = 2$, is rather complex. A precise characterization of this relation is important to understand how the truth conditions of cumulative sentences may be pragmatically strengthened by scalar implicatures. Shimada (to appear) gives two sufficient conditions for the entailment to hold, for the case $k = 2$. In Section 3, we reformulate Shimada's sufficient conditions and show that they form a complete set of inference rules.

In Section 4, we consider the case $k = 3$, and give some valid inference rules which will hopefully form part of a complete system of rules.

Let us write $RX_1 \dots X_k$ for the formula

$$\bigwedge_{i=1}^{k} X_i = \pi_i(R \cap (X_1 \times \cdots \times X_k)) \tag{5}$$

and $R(n_1, \dots, n_k)$ for the closed formula

$$\exists X_1 \dots \exists X_k \left(\bigwedge_{i=1}^{k} |X_i| = n_i \wedge RX_1 \dots X_k \right). \tag{6}$$

Since the truth conditions (3) can be equivalently expressed in the form (6) with $R = [\![\mathrm{V}]\!] \cap ([\![\mathrm{N}_1]\!] \times \cdots \times [\![\mathrm{N}_k]\!])$, the entailment relation between cumulative sentences (with the same choice of nouns and verbs) corresponds to the logical consequence relation between the closed formulas of the form (6) for various choices of $n_1, \dots, n_k$.

As usual, we write

$$R(n_1, \dots, n_k) \models R(m_1, \dots, m_k) \tag{7}$$

to mean $R(m_1, \dots, m_k)$ is a logical consequence of $R(n_1, \dots, n_k)$. Since (6) is evidently definable by an existential first-order formula with $n_1 + \cdots + n_k$ bound variables, the relation (7), understood as a $2k$-ary relation on the natural numbers, is decidable. This fact alone, however, does not suggest any efficient procedure to decide whether (7) is true.

Strictly speaking, logical consequence (in the model-theoretic sense) is a relation between closed formulas of a formal language, so when we write (7), we are using R as a predicate symbol, not as a k-ary relation (i.e., set of ordered k-tuples). A model for formulas of the form $R(n_1, \dots, n_k)$ is a structure $\mathfrak{M} = (M, R^{\mathfrak{M}})$, where M is a non-empty set and $R^{\mathfrak{M}}$ is a k-ary relation on M. The relation (7) holds if and only if every model of $R(n_1, \dots, n_k)$ is a model of $R(m_1, \dots, m_k)$. In this paper, we choose to be sloppy and will not make a clear distinction between syntax and semantics. This should not lead to any confusion.

2 Preliminaries

We employ the *projection* (π) and *selection* (σ) operators from relational database theory, using component numbers $1, 2, \ldots, k$ to pick out attributes of a k-ary relation (see Ullman, 1988). For example, if R is a ternary relation,

$$\pi_{1,2}(R) = \{\, (x, y) \mid (x, y, z) \in R \,\},$$
$$\sigma_{\$1 \neq a}(R) = \{\, (x, y, z) \in R \mid x \neq a \,\}.$$

Usually, the condition F in a selection operator σ_F must be a Boolean combination of equalities and inequalities, but in this paper we allow a selection operator of the form $\sigma_{\$i \in D}$.

When R is a binary relation, we also use standard notations like

- $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ for $\pi_1(R)$ and $\pi_2(R)$,
- R^{-1} for $\pi_{2,1}(R)$,
- $R^{-1}(b)$ for $\pi_1(\sigma_{\$2=b}(R))$,
- $R^{-1}(D)$ for $\pi_1(\sigma_{\$2 \in D}(R))$, etc.

If R is a binary relation, its *deterministic reduct* is[2]

$$d(R) = \{\, (x, y) \in R \mid \forall y'((x, y') \in R \to y' = y) \,\}.$$

Clearly, $d(R)$ is a partial function.

Lemma 1. $\sigma_{\$1 \in \mathrm{dom}(d(R))}(R) = d(R)$.

Proof. Clearly, $d(R) = \{\, (x, y) \in R \mid \forall y' \forall y''(((x, y') \in R \wedge (x, y'') \in R) \to y' = y'') \,\}$, and so $\mathrm{dom}(d(R)) = \{\, x \in \mathrm{dom}(R) \mid \forall y' \forall y''(((x, y') \in R \wedge (x, y'') \in R) \to y' = y'') \,\}$. It easily follows that $d(R) = \{\, (x, y) \in R \mid x \in \mathrm{dom}(d(R)) \,\}$. $\square$

Lemma 2. *If* $|\mathrm{dom}(R)| < |\mathrm{ran}(R)|$, *then* $|\mathrm{ran}(d(R))| < |\mathrm{dom}(R)|$.

Proof. Assume $|\mathrm{dom}(R)| < |\mathrm{ran}(R)|$. Since $d(R)$ is a partial function, $|\mathrm{dom}(d(R))| \geq |\mathrm{ran}(d(R))|$. This implies $R \neq d(R)$. Since $R = \sigma_{\$1 \in \mathrm{dom}(R)}(R)$, Lemma 1 implies $\mathrm{dom}(R) \supset \mathrm{dom}(d(R))$. So $|\mathrm{ran}(d(R))| \leq |\mathrm{dom}(d(R))| < |\mathrm{dom}(R)|$. $\square$

Lemma 3. $d(\sigma_{\$1 \in D}(R)) = \sigma_{\$1 \in D}(d(R))$.

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[2] This piece of terminology comes from descriptive complexity theory (Immerman, 1999), where the *deterministic transitive closure* of a binary relation is defined as the transitive closure of its deterministic reduct.

Proof. We have

$$\begin{aligned}
d(\sigma_{\$1\in D}(R)) &= \{\, (x,y) \in \sigma_{\$1\in D}(R) \mid \forall y'((x,y') \in \sigma_{\$1\in D}(R) \to y' = y) \,\} \\
&= \{\, (x,y) \in R \mid x \in D \wedge \forall y'(((x,y') \in R \wedge x \in D) \to y' = y) \,\} \\
&= \{\, (x,y) \in R \mid x \in D \wedge \forall y'((x,y') \in R \to y' = y) \,\} \\
&= \sigma_{\$1\in D}(\{\, (x,y) \in R \mid \forall y'((x,y') \in R \to y' = y) \,\}) \\
&= \sigma_{\$1\in D}(d(R)).
\end{aligned}$$

□

Lemma 4. *If* $\mathrm{dom}(R') = \mathrm{dom}(R)$ *and* $R' \subseteq R$*, then* $d(R') \supseteq d(R)$*.*

Proof. Suppose $\mathrm{dom}(R') = \mathrm{dom}(R)$ and $R' \subseteq R$. We have

$$\begin{aligned}
d(R') &= \{\, (x,y) \mid x \in \mathrm{dom}(R') \wedge \forall y'((x,y') \in R' \to y' = y) \,\} \\
&= \{\, (x,y) \mid x \in \mathrm{dom}(R) \wedge \forall y'((x,y') \in R' \to y' = y) \,\} \\
&\supseteq \{\, (x,y) \mid x \in \mathrm{dom}(R) \wedge \forall y'((x,y') \in R \to y' = y) \,\} \\
&= d(R).
\end{aligned}$$

□

It is easy to see that $RX_1 \dots X_k$, as defined by (5), is monotone in R: if $R \subseteq R'$, then $RX_1 \dots X_k$ implies $R'X_1 \dots X_k$. It follows that $RX_1 \dots X_k$ if and only if there exists some $R' \subseteq R$ such that $X_i = \pi_i(R')$ for $i = 1, \dots, k$. The following two lemmas are straightforward:

Lemma 5. *The following are equivalent:*

(i) $R(n_1, \dots, n_k)$.

(ii) $\exists R' \subseteq R \left(\bigwedge_{i=1}^{k} |\pi_i(R')| = n_i \right)$.

Lemma 6. *The following are equivalent:*

(i) $R(n_1, \dots, n_k) \models R(m_1, \dots, m_k)$.

(ii) $\bigwedge_{i=1}^{k} |\pi_i(R)| = n_i$ *implies* $R(m_1, \dots, m_k)$.

Lemma 7. *Let* R *be a* k*-ary relation. Let* $a \in \pi_i(R)$ *and* $R' = \sigma_{\$i\neq a}(R)$*. Then we have*

$$\begin{aligned}
&\pi_i(R') = \pi_i(R) - \{a\}, \\
&\pi_j(R') = \pi_j(R) - (d(\pi_{j,i}(R)))^{-1}(a) \quad \textit{for } j \neq i.
\end{aligned}$$

Proof. It is easy to see that $\pi_i(R') = \pi_i(R) - \{a\}$. Now let $j \neq i$ and suppose $y \in \pi_j(R')$. Then for some $(x_1, \ldots, x_k) \in R$, $x_j = y$ and $x_i \neq a$. Since $(y, x_i) \in \pi_{j,i}(R)$, we have $(y, a) \notin d(\pi_{j,i}(R))$. So $y \in \pi_j(R) - (d(\pi_{j,i}(R)))^{-1}(a)$. Conversely, suppose $y \in \pi_j(R) - (d(\pi_{j,i}(R)))^{-1}(a)$. This implies that there must be some $z \neq a$ such that $(y, z) \in \pi_{j,i}(R)$, which implies that for some $(x_1, \ldots, x_k) \in R$, $x_i = z$ and $x_j = y$. Since $x_i \neq a$, $(x_1, \ldots, x_k) \in R'$. So $y \in \pi_j(R')$. $\square$

Lemma 8. *If $A = \mathrm{dom}(R)$, $B = \mathrm{ran}(R)$, and $\varnothing \neq B' \subset B$, then there exists a $b \in B'$ such that $|(d(R))^{-1}(b)| \leq \lfloor (|A| - 1)/|B'| \rfloor$.*

Proof. Since $|(d(R))^{-1}(B')| = \sum_{y \in B'} |(d(R))^{-1}(y)|$, it suffices to show $(d(R))^{-1}(B') \subset A$. Suppose $(d(R))^{-1}(B') = A$. Then $\mathrm{dom}(d(R)) = A$ and $\mathrm{ran}(d(R)) = B'$. By Lemma 1, $R = \sigma_{\$1 \in A}(R) = \sigma_{\$1 \in \mathrm{dom}(d(R))}(R) = d(R)$. But $\mathrm{ran}(d(R)) = B' \neq B = \mathrm{ran}(R)$, a contradiction. $\square$

3 Binary Cumulative Sentences

In what follows, we use letters k, l, m, n as variables ranging over the set $\mathbb{N} - \{0\}$ of positive integers. The following lemmas are straightforward:

Lemma 9. *If $R(m, n) \models R(k, l)$, then one of the following holds:*

(i) *$m \geq n$ and $k \geq l$;*

(ii) *$m \leq n$ and $k \leq l$.*

Lemma 10. *If $R(m, n) \models R(k, l)$, then $m \geq k$ and $n \geq l$.*

The following lemma is Proposition 1 of Shimada (to appear):

Lemma 11. *If $m > n$, then $R(m, n) \models R(m - 1, n)$.*

Proof. Suppose $m > n$, and let R be a binary relation such that $A = \mathrm{dom}(R), B = \mathrm{ran}(R)$, and $|A| = m, |B| = n$. By Lemmas 5 and 6, it suffices to show the existence of an $R' \subseteq R$ such that $|\mathrm{dom}(R')| = m - 1$ and $|\mathrm{ran}(R')| = n$. Since $d(R^{-1})$ is a partial function from B to A and $|A| > |B|$, there must be an $a \in A$ such that $(d(R^{-1}))^{-1}(a) = \varnothing$. Let $R' = \sigma_{\$1 \neq a}(R)$. By Lemma 7, $\mathrm{dom}(R') = A - \{a\}$ and $\mathrm{ran}(R') = B$, and we are done. $\square$

The integer part $\lfloor m/n \rfloor$ of a fraction m/n is the quotient of m divided by n. We write $m \bmod n$ for the remainder of m divided by n. We always have $m = n \cdot \lfloor m/n \rfloor + (m \bmod n)$. The following lemma is adapted from Proposition 2 of Shimada (to appear):

Lemma 12. *If $m \geq n > 1$, then $R(m, n) \models R(m - \lfloor m/n \rfloor, n - 1)$.*

Proof. Suppose $m \geq n > 1$, and let R be a binary relation such that $A = \mathrm{dom}(R), B = \mathrm{ran}(R)$, and $|A| = m, |B| = n$. By Lemma 6, it suffices to show $R(m - \lfloor m/n \rfloor, n-1)$. Since $m = |A| \geq |(d(R))^{-1}(B)| = \sum_{y \in B} |(d(R))^{-1}(y)|$, there must be a $b \in B$ such that $|(d(R))^{-1}(b)| \leq \lfloor m/n \rfloor$. Let $R' = \sigma_{\$2 \neq b}(R)$. By Lemma 7, $\mathrm{dom}(R') = A - (d(R))^{-1}(b)$ and $\mathrm{ran}(R') = B - \{b\}$. Let $m' = |A - (d(R))^{-1}(b)|$. Then $m' \geq m - \lfloor m/n \rfloor$. By Lemma 5, we have $R(m', n-1)$.

Since $m \geq n$, we have

$$\begin{aligned} m - \lfloor m/n \rfloor &\geq n \cdot \lfloor m/n \rfloor - \lfloor m/n \rfloor \\ &= (n-1) \cdot \lfloor m/n \rfloor \\ &\geq n - 1. \end{aligned}$$

So Lemma 11 implies $R(m - \lfloor m/n \rfloor, n-1)$. □

It is evident that the symmetric variants of Lemmas 11 and 12 hold as well: $m < n$ implies $R(m,n) \models R(m, n-1)$ and $1 < m \leq n$ implies $R(m,n) \models R(m-1, n - \lfloor n/m \rfloor)$. Combining all these, we see that $R(m,n) \models R(m-1, n-\lfloor n/m \rfloor)$ holds whenever $m > 1$, and symmetrically, $R(m,n) \models R(m - \lfloor m/n \rfloor, n-1)$ holds whenever $n > 1$.

Let us write $R(m,n) \vdash R(k,l)$ if $R(k,l)$ can be deduced from $R(m,n)$ using the following rules of inference:

$$\frac{R(m,n) \quad m > 1}{R(m-1, n - \lfloor n/m \rfloor)} \text{ (R2–1)} \qquad \frac{R(m,n) \quad n > 1}{R(m - \lfloor m/n \rfloor, n-1)} \text{ (R2–2)}$$

Theorem 13 (Soundness)**.** *If $R(m,n) \vdash R(k,l)$, then $R(m,n) \models R(k,l)$.*

Lemma 14 (Characterization)**.** *Let $m \geq n$. The following conditions are equivalent:*

(i) $R(m,n) \models R(k,l)$.

(ii) $k \leq m \wedge l \leq n \wedge l \leq k \leq l \cdot \lfloor m/n \rfloor + \min(m \bmod n, l)$.

(iii) $R(m,n) \vdash R(k,l)$.

Proof. (i) ⇒ (ii). By Lemmas 9 and 10, we only need to show

$$k \leq l \cdot \lfloor m/n \rfloor + \min(m \bmod n, l). \tag{8}$$

Let $A = \{a_0, \ldots, a_{m-1}\}$, $B = \{b_0, \ldots, b_{n-1}\}$, and consider the binary relation

$$R = \{ (a_i, b_j) \mid j = i \bmod n \}.$$

Clearly, $\mathrm{dom}(R) = A$ and $\mathrm{ran}(R) = B$, so $R(m,n)$ holds. By the assumption (i), $R(k,l)$. Let $A' \subseteq A, B' \subseteq B$ be such that $|A'| = k$, $|B'| = l$, and $RA'B'$. We must have

$$A' \subseteq \bigcup_{b_j \in B'} \{ a_i \mid j = i \bmod n \},$$

so

$$k \leq \sum_{b_j \in B'} |\{\, i \mid j = i \bmod n \,\}|. \tag{9}$$

We distinguish two cases.

Case 1. n divides m, i.e., $m \bmod n = 0$. In this case, it is clear that

$$|\{\, i \mid j = i \bmod n \,\}| = m/n = \lfloor m/n \rfloor$$

holds for each $j = 0, \ldots, n-1$. By (9),

$$k \leq l \cdot \lfloor m/n \rfloor,$$

which implies (8).

Case 2. n does not divide m, i.e., $m \bmod n \geq 1$. In this case,

$$|\{\, i \mid j = i \bmod n \,\}| = \begin{cases} \lfloor m/n \rfloor + 1 & \text{if } 0 \leq j < m \bmod n, \\ \lfloor m/n \rfloor & \text{if } m \bmod n \leq j \leq n-1. \end{cases}$$

Then (8) easily follows from (9).

(ii) $\Rightarrow$ (iii). By (R2–1), it suffices to prove

$$R(m,n) \vdash R(l \cdot \lfloor m/n \rfloor + \min(m \bmod n, l), l). \tag{10}$$

We show this by induction on $n - l$. If $n - l = 0$, then $l = n > m \bmod n$, so $l \cdot \lfloor m/n \rfloor + \min(m \bmod n, l) = n \cdot \lfloor m/n \rfloor + (m \bmod n) = m$, and (10) becomes $R(m,n) \vdash R(m,n)$, which holds trivially.

Now suppose $n - l \geq 1$. The induction hypothesis says

$$R(m,n) \vdash R((l+1) \cdot \lfloor m/n \rfloor + \min(m \bmod n, l+1), l+1).$$

Let

$$k' = (l+1) \cdot \lfloor m/n \rfloor + \min(m \bmod n, l+1),$$

so that the right-hand side of the induction hypothesis becomes $R(k', l+1)$. Since $l + 1 > 1$, an application of the rule (R2–2) gives

$$R(k', l+1) \vdash R(k' - \lfloor k'/(l+1) \rfloor, l).$$

Combining this with the induction hypothesis, we get

$$R(m,n) \vdash R(k' - \lfloor k'/(l+1) \rfloor, l). \tag{11}$$

Case 1. $m \bmod n < l + 1$. Then $k' = (l+1) \cdot \lfloor m/n \rfloor + (m \bmod n)$. Now

$$\begin{aligned} \frac{k'}{l+1} &= \frac{(l+1) \cdot \lfloor m/n \rfloor + (m \bmod n)}{l+1} \\ &= \lfloor m/n \rfloor + \frac{m \bmod n}{l+1}. \end{aligned}$$

It follows that

$$\lfloor k'/(l+1) \rfloor = \lfloor m/n \rfloor,$$

since $m \bmod n < l+1$. So

$$\begin{aligned} k' - \lfloor k'/(l+1) \rfloor &= (l+1) \cdot \lfloor m/n \rfloor + (m \bmod n) - \lfloor m/n \rfloor \\ &= l \cdot \lfloor m/n \rfloor + (m \bmod n). \end{aligned} \tag{12}$$

Since $m \bmod n < l+1$, $m \bmod n \leq l$ and $\min(m \bmod n, l) = m \bmod n$. So (10) becomes

$$R(m,n) \vdash R(l \cdot \lfloor m/n \rfloor + (m \bmod n), l),$$

which we can obtain from (11) by substituting (12) for $k' - \lfloor k'/(l+1) \rfloor$.

Case 2. $m \bmod n \geq l+1$. Then $k' = (l+1) \cdot \lfloor m/n \rfloor + l + 1 = (l+1) \cdot (\lfloor m/n \rfloor + 1)$. So

$$\begin{aligned} k' - \lfloor k'/(l+1) \rfloor &= (l+1) \cdot (\lfloor m/n \rfloor + 1) - (\lfloor m/n \rfloor + 1) \\ &= l \cdot (\lfloor m/n \rfloor + 1) \\ &= l \cdot \lfloor m/n \rfloor + l. \end{aligned} \tag{13}$$

Since $m \bmod n \geq l+1 > l$, we have $\min(m \bmod n, l) = l$, and (10) becomes

$$R(m,n) \vdash R(l \cdot \lfloor m/n \rfloor + l, l),$$

which we can obtain from (11) by substituting (13) for $k' - \lfloor k'/(l+1) \rfloor$.

(iii) $\Rightarrow$ (i). By Theorem 13. $\square$

Theorem 15 (Completeness). *$R(m,n) \vdash R(k,l)$ if and only if $R(m,n) \models R(k,l)$.*

Proof. Immediate form Lemma 14 and its symmetric variant for $m \leq n$. $\square$

Figure 2 shows a Hasse diagram for the partial order representing the entailment relation between binary cumulative sentences $R(m,n)$ with $m, n \leq 8$.

4 Ternary Cumulative Sentences

The entailment relation between ternary cumulative sentences seems to be much more complicated than the binary case, and we only have some partial results so far.

Lemma 16. *Suppose $m \geq n$. Then*

$$\{R(m,n,p)\} \cup \{\, \neg R(m-i, n-j, p) \mid 0 \leq i < j < n \,\}$$

is satisfiable.

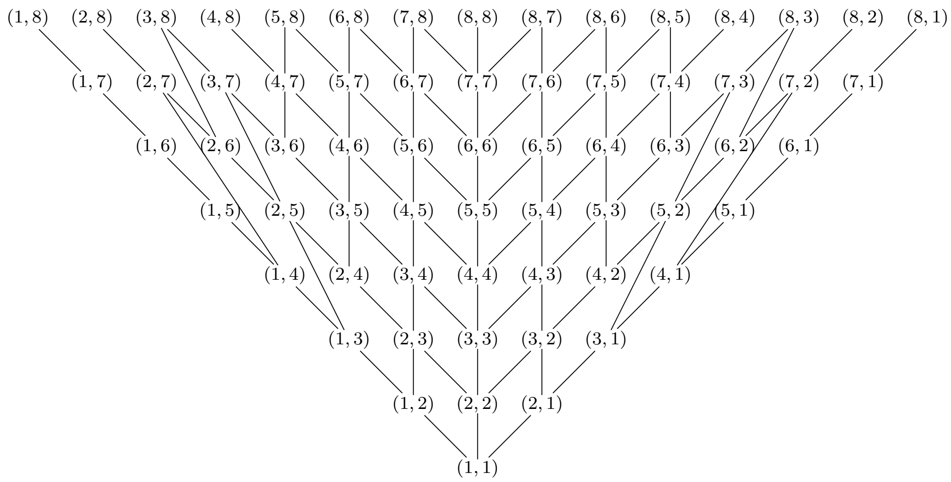

Figure 2: A Hasse diagram of entailment between binary cumulative sentences (from Shimada, to appear).

Proof. This set is satisfied by any binary relation R such that $R(m, n, p)$ and $\pi_{1,2}(R)$ is a function. $\square$

Lemma 17. *Suppose $m \leq n + p - 2$. Then*

$$\{R(m, n, p)\} \cup \{\, \neg R(m', n, p) \mid m' < m \,\}$$

is satisfiable.

Proof. Let $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}, C = \{c_1, \ldots, c_p\}$, and

$$\begin{aligned} R = \; & \{\, (a_{\min(j,m)}, b_j, c_p) \mid 1 \leq j \leq n - 1 \,\} \cup \\ & \{\, (a_{\min(n-1+k,m)}, b_n, c_k) \mid 1 \leq k \leq p - 1 \,\}. \end{aligned}$$

Then $\pi_1(R) = A, \pi_2(R) = B, \pi_3(R) = C$, so $R(m, n, p)$. Note that for all $x \in A$, either $(d(\pi_{2,1}(R)))^{-1}(x) \neq \varnothing$ or $(d(\pi_{3,1}(R)))^{-1}(x) \neq \varnothing$. Suppose $m' < m$ and $R(m', n, p)$, so that $\pi_1(R') \subset A, \pi_2(R') = B, \pi_3(R') = C$ for some $R' \subseteq R$. Let $a \in A - \pi_1(R')$. Then $\sigma_{\$1 \neq a}(R) \supseteq R'$, so $\pi_2(\sigma_{\$1 \neq a}(R)) = B$ and $\pi_3(\sigma_{\$1 \neq a}(R)) = C$. Lemma 7 then implies $(d(\pi_{2,1}(R)))^{-1}(a) = (d(\pi_{3,1}(R)))^{-1}(a) = \varnothing$, a contradiction. $\square$

Proposition 18. *The following inference rule is valid:*

$$\frac{R(m, n, p) \quad m > n \quad m > p \quad m > n + p - 2}{R(m - 1, n, p)} \text{ (R3–1)}$$

Proof. Assume $m > n, m > p, m > n + p - 2$, and let R be a ternary relation such that $A = \pi_1(R), B = \pi_2(R), C = \pi_3(R)$ and $|A| = m, |B| = n, |C| = p$. We show that there exists an $R' \subseteq R$ such that $|\pi_1(R')| = m - 1, |\pi_2(R')| = n, |\pi_3(R')| = p$. Let $S = d(\pi_{2,1}(R))$ and $T = d(\pi_{3,1}(R))$. Since $m > n$ and $m > p$, Lemma 2 implies $|\mathrm{ran}(S)| \leq n - 1$ and $|\mathrm{ran}(T)| \leq p - 1$. Since $m > n + p - 2$, it follows that there is an $a \in A - (\mathrm{ran}(S) \cup \mathrm{ran}(T))$. Let $R' = \sigma_{\$1 \neq a}(R)$. Since $S^{-1}(a) = T^{-1}(a) = \varnothing$, we get $\pi_1(R') = A - \{a\}$, $\pi_2(R') = B$, and $\pi_3(R') = C$ by Lemma 7. □

Lemma 19. *Suppose* $m - 1 \geq 2(n - p + 1)$ *and* $p \geq 2$. *Then*

$$\{R(m, n, p)\} \cup \{\, \neg R(m - 1, n', p) \mid n' < n \,\}$$

is satisfiable.

Proof. If $n \leq p$, it is easy to see that the conclusion of the lemma holds, so let us suppose $n > p$.

Let $A = \{a_0, \ldots, a_{m-1}\}, B = \{b_0, \ldots, b_{n-1}\}, C = \{c_0, \ldots, c_{p-1}\}$, and define

$$\begin{aligned} R = \{\ (a_i, b_j, c_k) \mid\ & (i \leq m - 2 \wedge j = i \bmod (n - p + 1) \wedge k = 0) \vee \\ & (i = m - 1 \wedge j = n - p + k \wedge k \geq 1) \,\}. \end{aligned}$$

It is clear that $A = \pi_1(R), B = \pi_2(R), C = \pi_3(R)$, so $R(m, n, p)$ holds. For each $j \leq n - p$,

$$\{a_j, a_{n-p+1+j}\} \subseteq (d(\pi_{1,2}(R)))^{-1}(b_j),$$

and for each $j \geq n - p + 1$,

$$c_{j-(n-p)} \in (d(\pi_{3,2}(R)))^{-1}(b_j).$$

This property ensures that $R(m - 1, n', p)$ does not hold for any $n' < n$. □

Proposition 20. *Suppose* $n-1 \geq 2(m-p+1)$, $m \geq n$, *and* $p \geq 2$. *Then* $R(m, n, p) \models R(m', n', p)$ *implies* $m' = m$ *and* $n' = n$.

Proof. Suppose $R(m, n, p) \models R(m', n', p)$. By Lemma 16,

$$m - m' \geq n - n'.$$

In particular, $m' = m$ implies $n' = n$.

Assume $m' < m$. Since

$$\begin{aligned} m &\leq \frac{n-1}{2} + p - 1 \\ &\leq n + p - 2, \end{aligned}$$

Lemma 17 applies, so $n' < n$.

Let R be a ternary relation such that $A = \pi_1(R), B = \pi_2(R), C = \pi_3(R)$, and $|A| = m, |B| = n, |C| = p$. By Lemmas 5 and 6, there exist $R' \subseteq R$ and A', B' such that $\pi_1(R') = A', \pi_2(R') = B', \pi_3(R') = C$, and $|A'| = m', |B'| = n'$. Let $b_1, \ldots, b_{n-n'}$ be the elements of $B - B'$, and for each $i = 1, \ldots, n - n'$, pick a_i and c_i such that $(a_i, b_i, c_i) \in R$. Let

$$R'' = R' \cup \{(a_1, b_1, c_1), \ldots, (a_{n-n'-1}, b_{n-n'-1}, c_{n-n'-1})\}.$$

Then $R'' \subseteq R$, and it is easy to see

$$\begin{aligned} |\pi_1(R'')| &\leq m' + n - n' - 1, \\ |\pi_2(R'')| &= n - 1, \\ |\pi_3(R'')| &= p. \end{aligned}$$

Since $m - m' \geq n - n'$, we have

$$\begin{aligned} m' + n - n' - 1 &\leq m' + m - m' - 1 \\ &= m - 1. \end{aligned}$$

This shows that $R(m'', n-1, p)$ holds for some $m'' < m$ whenever $R(m, n, p)$ holds. On the other hand, since $n - 1 \geq 2(m - p + 1)$ and $p \geq 2$, a variant of Lemma 19 says

$$\{R(m, n, p)\} \cup \{\, \neg R(m', n - 1, p) \mid m' < m \,\}$$

is satisfiable, a contradiction. $\square$

For example, $(m, n, p) = (15, 13, 10)$ satisfies the conditions of Proposition 20, so $R(15, 13, 10) \models R(m', n', 10)$ only if $m' = 15$ and $n' = 13$.

Proposition 21. *The following inference rule is valid:*

$$\frac{R(m,n,p) \quad 2(n-p+1) \geq m > p \quad 2(m-p+1) \geq n > p \quad p \geq 2}{R(m-1, n-1, p)} \quad \text{(R3–2)}$$

Proof. Suppose $2(n-p+1) \geq m > p$, $2(m-p+1) \geq n > p$, and $p \geq 2$. Let R be a ternary relation such that $A = \pi_1(R), B = \pi_2(R), C = \pi_3(R)$, and $|A| = m, |B| = n, |C| = p$. We show that there is an $R' \subseteq R$ such that $|\pi_1(R')| = m - 1$, $|\pi_2(R')| = n - 1$, and $|\pi_3(R')| = p$.

Since $n > p$, Lemma 2 implies $|\mathrm{ran}(d(\pi_{3,2}(R)))| \leq p - 1$. So $|B - \mathrm{ran}(d(\pi_{3,2}(R)))| \geq n - p + 1$. Let B' be a subset of $B - \mathrm{ran}(d(\pi_{3,2}(R)))$ with $|B'| = n - p + 1$. Since $p \geq 2$, $B' \subset B$. By Lemma 8, there is a $b \in B'$ such that $|(d(\pi_{1,2}(R)))^{-1}(b)| \leq \lfloor (m-1)/(n-p+1) \rfloor$. Since $2(n - p + 1) \geq m$, we have $\lfloor (m-1)/(n-p+1) \rfloor \leq 1$, so $(d(\pi_{1,2}(R)))^{-1}(b)$ is either empty or a singleton. Let

$$R_1 = \sigma_{\$2 \neq b}(R).$$

Since $b \notin \operatorname{ran}(d(\pi_{3,2}(R)))$, we have $(d(\pi_{3,2}(R)))^{-1}(b) = \varnothing$. By Lemma 7, $\pi_2(R_1) = B - \{b\}$ and $\pi_3(R_1) = C$.

Case 1. $(d(\pi_{1,2}(R)))^{-1}(b) = \{a\}$ for some $a \in A$. By Lemma 7, we have $\pi_1(R_1) = A - \{a\}$, and we are done.

Case 2. $(d(\pi_{1,2}(R)))^{-1}(b) = \varnothing$. By Lemma 7, we have $\pi_1(R_1) = A$. By Lemma 2, $|\operatorname{ran}(d(\pi_{3,1}(R_1)))| \leq p-1$, so $|A - \operatorname{ran}(d(\pi_{3,1}(R_1)))| \geq m-p+1$. Let A' be a subset of $A - \operatorname{ran}(d(\pi_{3,1}(R_1)))$ with $|A'| = m-p+1$. By Lemma 8, we can find an $a' \in A'$ such that $|(d(\pi_{2,1}(R)))^{-1}(a')| \leq \lfloor (n-1)/(m-p+1) \rfloor \leq 1$. So $(d(\pi_{2,1}(R)))^{-1}(a')$ is either empty or a singleton. Since $a' \notin \operatorname{ran}(d(\pi_{3,1}(R_1)))$, we have $(d(\pi_{3,1}(R_1)))^{-1}(a') = \varnothing$.

Case 2.1. $(d(\pi_{2,1}(R)))^{-1}(a') = \{b'\}$ for some $b' \in B$. Let

$$R_2 = \sigma_{\$1 \neq a'}(R).$$

We know that $(d(\pi_{3,1}(R_1)))^{-1}(a') = \varnothing$. But since $\operatorname{dom}(\pi_{3,1}(R_1)) = \pi_3(R_1) = C = \pi_3(R) = \operatorname{dom}(\pi_{3,1}(R))$, Lemma 4 implies $d(\pi_{3,1}(R_1)) \supseteq d(\pi_{3,1}(R))$. It follows that $(d(\pi_{3,1}(R)))^{-1}(a') = \varnothing$ as well. By Lemma 7, we have $\pi_1(R_2) = A - \{a'\}$, $\pi_2(R_2) = B - \{b'\}$, and $\pi_3(R_2) = C$.

Case 2.2. $(d(\pi_{2,1}(R)))^{-1}(a') = \varnothing$. Since it is easy to see $\pi_{2,1}(R_1) = \pi_{2,1}(\sigma_{\$2 \neq b}(R)) = \sigma_{\$1 \neq b}(\pi_{2,1}(R))$, Lemma 3 implies $d(\pi_{2,1}(R_1)) = \sigma_{\$1 \neq b}(d(\pi_{2,1}(R)))$. This means that $(d(\pi_{2,1}(R_1)))^{-1}(x) = (d(\pi_{2,1}(R)))^{-1}(x) - \{b\}$ for every $x \in A$. In particular, $(d(\pi_{2,1}(R_1)))^{-1}(a') = \varnothing - \{b\} = \varnothing$. Let

$$R_3 = \sigma_{\$1 \neq a'}(R_1).$$

Since we also know that $(d(\pi_{3,1}(R_1)))^{-1}(a') = \varnothing$, Lemma 7 gives $\pi_1(R_3) = \pi_1(R_1) - \{a'\} = A - \{a'\}$, $\pi_2(R_3) = \pi_2(R_1) = B - \{b\}$, and $\pi_3(R_3) = \pi_3(R_1) = C$. $\square$

Proposition 22. *The following inference rule is valid:*

$$\frac{R(m,n,p) \quad m - \lfloor (m-1)/(n-p+1) \rfloor \geq n+p-3 \quad n > p \geq 2}{R(m - \lfloor (m-1)/(n-p+1) \rfloor, n-1, p)} \text{ (R3–3)}$$

Proof. Suppose that $m - \lfloor (m-1)/(n-p+1) \rfloor \geq n+p-3$ and $n > p \geq 2$. Let R be a ternary relation such that $A = \pi_1(R), B = \pi_2(R), C = \pi_3(R)$, and $|A| = m, |B| = n, |C| = p$. We show that $R(m - \lfloor (m-1)/(n-p+1) \rfloor, n-1, p)$ holds.

Let $S = d(\pi_{3,2}(R))$ and $T = d(\pi_{1,2}(R))$. Since $n > p$, $|B - \operatorname{ran}(S)| \geq n-p+1$ by Lemma 2. Since $p \geq 2$, $n-p+1 < n$. By Lemma 8, there is a $b \in B - \operatorname{ran}(S)$ such that $|T^{-1}(b)| \leq \lfloor (m-1)/(n-p+1) \rfloor$. Since $b \notin \operatorname{ran}(S)$, $S^{-1}(b) = \varnothing$. Let $R' = \sigma_{\$2 \neq b}(R)$. By Lemma 7, $\pi_1(R') = A - T^{-1}(b), \pi_2(R') = B - \{b\}, \pi_3(R') = C$, so $R(m', n-1, p)$ for some $m' \geq m - \lfloor (m-1)/(n-p+1) \rfloor$. Since $m - \lfloor (m-1)/(n-p+1) \rfloor \geq n+p-3 \geq (n-1)+p-2$, Lemma 18 implies $R(m - \lfloor (m-1)/(n-p+1) \rfloor, n-1, p)$. $\square$

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