Completeness and Decidability of the Mixed Style of Inference with Composition

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Abstract

I present a complete calculus for mixed inference (van Benthem 1991) with composition and prove that it has the finite model property and is therefore decidable. I also present a variant of the calculus complete with respect to deterministic models, and mention the completeness and (un)decidability of other styles of inference involving composition, including dynamic inference (van Benthem 1991).

A recent trend in one corner of logic is to regard the meaning of a sentence as a relation on information states. As discussed in van Benthem 1991, this 'dynamic' perspective gives rise to a number of new conceptions of inference, allowing different answers to the question of what it means for conclusion C to follow from premises $P_1, \ldots, P_n$. One such dynamic notion of inference is what van Benthem (1991) calls *mixed inference*, which is of particular interest for its connection to Veltman's (1991) update semantics. In this paper, we investigate mixed inference with respect to such standard logical questions as axiomatizability and decidability. Other dynamic styles of inference will be discussed briefly in the last section of the paper.

1 Van Benthem's Mixed Inference

1.1 Calculus M

In his 1991 paper, van Benthem introduces the following calculus **M**, to capture the general properties of the style of inference that he calls *mixed inference*. **M** is a calculus for deriving a sequent from a set of sequents. In **M**, a *sequent* is an expression of the form $X \Rightarrow d$, where X is a finite sequence of atomic formulas, and d is an atomic formula. In what follows, p, q, c, d (with or without subscripts) range over atomic formulas, and X, Y, Z, W, V (with or without subscripts) range over finite sequences of formulas.

Calculus M.

- Axioms: **M** has no axioms.
- Rules of Inference: **M** has two rules of inference:[1]

Left Monotonicity $$\frac{X \Rightarrow d}{p\, X \Rightarrow d}$$

Left Cut $$\frac{X \Rightarrow c \quad X\, c\, Y \Rightarrow d}{X\, Y \Rightarrow d}$$

We write $\Gamma \vdash_{\mathbf{M}} \mathcal{I}$ if Γ is a finite set of sequents and $\mathcal{I}$ is a sequent derivable from sequents in Γ using the two rules of inference of **M**.

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[1] In van Benthem 1991, the following version of Left Cut was used:

Left Cut $$\frac{X \Rightarrow c \quad Y\, X\, c\, Z \Rightarrow d}{Y\, X\, Z \Rightarrow d}$$

In the presence of Left Monotonicity, the two versions of Left Cut are equivalent.

1.2 Semantics

The intended semantics for **M** is as follows. A model for **M** is a structure $M = \langle |M|, R_0, R_1, R_2, \ldots \rangle$, where $|M|$ is a non-empty set and each R_i is a binary relation on $|M|$.

- The interpretation $[\![q_i]\!]_M$ of the i-th atomic formula q_i is R_i.
- $M \models p_1 \ldots p_n \Rightarrow d$ if and only if $\mathrm{range}([\![p_1]\!]_M \circ \cdots \circ [\![p_n]\!]_M) \subseteq \mathrm{fix}([\![d]\!]_M)$, where $\mathrm{fix}(R) = \{\, x \mid \langle x, x \rangle \in R \,\}$.

We write $M \models \Gamma$ if $M \models \mathcal{I}$ for all $\mathcal{I}$ in Γ.

- $\Gamma \models \mathcal{I}$ iff for all M, $M \models \Gamma$ implies $M \models \mathcal{I}$.

M is complete with respect to the intended semantics in the sense that $\Gamma \vdash_{\mathbf{M}} \mathcal{I}$ if and only if $\Gamma \models \mathcal{I}$. This is proved in van Benthem 1991, by showing a method of constructing for any finite set Γ of sequents a canonical model M_Γ such that for every sequent $\mathcal{I}$, $M_\Gamma \models \mathcal{I}$ if and only if $\Gamma \vdash_{\mathbf{M}} \mathcal{I}$.

1.3 Decidability

It is easy to see that **M** has the finite model property; i.e., **M** is complete with respect to the class of finite models. This is so because the definition of $M \models \mathcal{I}$ translates into a universal first-order sentence without function symbols. Suppose $M \models \Gamma$ and $M \not\models p_1 \ldots p_n \Rightarrow d$, so that there are $x_0, \ldots, x_n$ such that $\langle x_0, x_1 \rangle \in [\![p_1]\!]_M, \ldots, \langle x_{n-1}, x_n \rangle \in [\![p_n]\!]_M$ and $\langle x_n, x_n \rangle \notin [\![d]\!]_M$. Then the finite submodel M_0 of M such that $|M_0| = \{x_0, \ldots, x_n\}$ has $M_0 \models \Gamma$ and $M_0 \not\models p_1 \ldots p_n \Rightarrow d$. That **M** has the finite model property implies that it is decidable.

Grigori Mints (p.c.) has shown the decidability of **M** using an equivalent natural deduction type calculus **NDM**, for which he proves a normalization theorem. (See my notes Kanazawa 1993a, 1993b.)

2 Adding Connectives

The language of **M** has no connective, so **M** does not deal with complex formulas. It would be interesting to see how **M** can be extended to languages with various connectives. Here we consider two conjunctions, $\cap$ and $\bullet$, which are interpreted as intersection and composition, respectively. Let M be a model in the above sense. The interpretation of a complex formula is given as follows. (Below, P, C, D (with or without subscripts) range over (possibly complex) formulas.)

- $[\![D_1 \cap D_2]\!]_M = [\![D_1]\!]_M \cap [\![D_2]\!]_M$.
- $[\![D_1 \bullet D_2]\!]_M = [\![D_1]\!]_M \circ [\![D_2]\!]_M$.

A sequent is now an expression of the form $X \Rightarrow D$, where X is a finite sequence of (possibly complex) formulas, and D is a (possibly complex) formula. The definition of truth and semantic consequence remains the same, except that we are now dealing with complex formulas as well as atomic ones.

- $M \models P_1 \ldots P_n \Rightarrow D$ if and only if $\mathrm{range}([\![P_1]\!]_M \circ \cdots \circ [\![P_n]\!]_M) \subseteq \mathrm{fix}([\![D]\!]_M)$.
- $\Gamma \models \mathcal{I}$ iff for all M, $M \models \Gamma$ implies $M \models \mathcal{I}$.

The problem now is to find a complete set of rules governing the newly introduced connectives.

2.1 Intersection

As usual, intersection is the easier one to deal with. It is quite straightforward to extend **M** to the language with $\cap$ as its only connective. (Below, $P[C]$ denotes a formula with a specified subformula occurrence C.)

Calculus M($\cap$).

- Axioms: **M**($\cap$) has no axioms.
- Rules of Inference: **M**($\cap$) has the following rules of inference:

Left Monotonicity $$\frac{X \Rightarrow D}{P\,X \Rightarrow D}$$

Left Cut $$\frac{X \Rightarrow C \quad X\,C\,Y \Rightarrow D}{X\,Y \Rightarrow D}$$

($\cap \Rightarrow$) $$\frac{X\,P_i\,Y \Rightarrow D}{X\,P_1 \cap P_2\,Y \Rightarrow D} \quad i = 1, 2$$

($\Rightarrow \cap_1$) $$\frac{X \Rightarrow D_1 \quad X \Rightarrow D_2}{X \Rightarrow D_1 \cap D_2}$$

($\Rightarrow \cap_2$) $$\frac{X \Rightarrow D_1 \cap D_2}{X \Rightarrow D_i} \quad i = 1, 2$$

(Assoc. $\cap \Rightarrow$) $$\frac{X\,P[((C_1 \cap C_2) \cap C_3)]\,Y \Rightarrow D}{X\,P[(C_1 \cap (C_2 \cap C_3))]\,Y \Rightarrow D} \quad \downarrow\uparrow \text{ both ways}$$

(Perm. $\cap \Rightarrow$) $$\frac{X\,P[(C_1 \cap C_2)]\,Y \Rightarrow D}{X\,P[(C_2 \cap C_1)]\,Y \Rightarrow D}$$

(Contr. $\cap \Rightarrow$) $$\frac{X\,P[(C \cap C)]\,Y \Rightarrow D}{X\,P[C]\,Y \Rightarrow D}$$

The completeness of **M**($\cap$) can be shown by a minor modification of van Benthem's (1991) construction. The finite model property of **M**($\cap$) is also obvious.

2.2 Composition

Let us now consider the language with $\bullet$ as its only connective. Although it is not immediately obvious, the following set of rules turns out to constitute a complete calculus for this language.

Calculus M($\bullet$).

- Axioms: **M**($\bullet$) has no axioms.
- Rules of Inference: **M**($\bullet$) has the following rules of inference:

Left Monotonicity $$\frac{X \Rightarrow D}{P\,X \Rightarrow D}$$

Left Cut $$\frac{X \Rightarrow C \quad X\,C\,Y \Rightarrow D}{X\,Y \Rightarrow D}$$

($\bullet \Rightarrow_1$) $$\frac{X\,P_1\,P_2\,Y \Rightarrow D}{X\,P_1 \bullet P_2\,Y \Rightarrow D}$$

($\bullet \Rightarrow_2$) $$\frac{X\,P_1 \bullet P_2\,Y \Rightarrow D}{X\,P_1\,P_2\,Y \Rightarrow D}$$

$$(\Rightarrow \bullet_1) \qquad \frac{X \Rightarrow D_1 \quad X \Rightarrow D_2}{X \Rightarrow D_1 \bullet D_2}$$

$$(\Rightarrow \bullet_2) \qquad \frac{X\ D_1 \Rightarrow D_2 \quad X \Rightarrow D_1 \bullet D_3}{X \Rightarrow (D_1 \bullet D_2) \bullet D_3}$$

$$(\Rightarrow \text{Assoc.}\ \bullet) \qquad \frac{X \Rightarrow D[((C_1 \bullet C_2) \bullet C_3)]}{\overline{X \Rightarrow D[(C_1 \bullet (C_2 \bullet C_3))]}} \ \downarrow\uparrow \text{ both ways}$$

Moreover, $\mathbf{M}(\bullet)$ also enjoys the finite model property, and is therefore decidable. This requires a slightly more elaborate argument than before.[2]

We prove the completeness and finite model property of $\mathbf{M}(\bullet)$ in the following three sections. Here, let us note

Lemma 2.1 *The following rule is derivable in* $\mathbf{M}(\bullet)$*:*

$$(\Rightarrow \bullet_3) \qquad \frac{X\ D_1 \Rightarrow D_2 \quad X \Rightarrow D_1}{X \Rightarrow D_1 \bullet D_2}$$

PROOF.

$$\dfrac{X \Rightarrow D_1 \qquad \dfrac{X \Rightarrow D_1 \quad X\ D_1 \Rightarrow D_2}{X \Rightarrow D_2}\,\text{Left Cut}}{X \Rightarrow D_1 \bullet D_2}\,(\Rightarrow \bullet_1)$$

■

Remark $(\Rightarrow \bullet_1)$ and $(\Rightarrow \bullet_3)$ can be thought of as special cases of $(\Rightarrow \bullet_2)$, where D_1 and D_3 are 'empty', respectively.

3 Calculus $\mathbf{M}\mu$

In proving the completeness and finite model property of $\mathbf{M}(\bullet)$, it is convenient to work with an equivalent calculus $\mathbf{M}\mu$ with multiple succedents, whose language has no connective. An $\mathbf{M}\mu$ sequent is of the form $X \Rightarrow Y$, where X is a finite sequence of atomic formulas and Y is a non-empty finite sequence of atomic formulas.

Calculus $\mathbf{M}\mu$.

- Axioms: $\mathbf{M}\mu$ has no axioms.
- Rules of Inference: $\mathbf{M}\mu$ has three rules of inference:

$$\text{Left Monotonicity} \qquad \frac{X \Rightarrow Y}{p\ X \Rightarrow Y}$$

$$\text{Left Cut} \qquad \frac{X \Rightarrow Y \quad X\ Y\ Z \Rightarrow W}{X\ Z \Rightarrow W}$$

$$(\Rightarrow \bullet) \qquad \frac{X\ Y \Rightarrow Z \quad X \Rightarrow Y\ W}{X \Rightarrow Y\ Z\ W}$$

The definition of truth for $\mathbf{M}\mu$ sequents is as follows:

- $M \models p_1 \dots p_n \Rightarrow d_1 \dots d_m$ if and only if $\text{range}([\![p_1]\!]_M \circ \cdots \circ [\![p_n]\!]_M) \subseteq \text{fix}([\![d_1]\!]_M \circ \cdots \circ [\![d_m]\!]_M)$.

[2] That $\mathbf{M}(\bullet)$ is decidable is perhaps mildly surprising, as it is easy to show that the corresponding calculus for *dynamic inference* (van Benthem 1991) with composition is undecidable. See Section 8.

If $\mathcal{I}$ is an $\mathbf{M}(\bullet)$ sequent, let $\mathcal{I}^\sharp$ be the $\mathbf{M}\mu$ sequent which results from erasing all occurrences of $\bullet$ and parentheses in $\mathcal{I}$.

Lemma 3.1 *$M \models \mathcal{I}$ if and only if $M \models \mathcal{I}^\sharp$.*

Lemma 3.2 *$\Gamma \vdash_{\mathbf{M}(\bullet)} \mathcal{I}$ if and only if $\Gamma^\sharp \vdash_{\mathbf{M}\mu} \mathcal{I}^\sharp$, where $\Gamma^\sharp = \{\, \mathcal{J}^\sharp \mid \mathcal{J} \in \Gamma \,\}$.*

By the above lemmas, to show the completeness and finite model property of $\mathbf{M}(\bullet)$, it is enough to show the completeness and finite model property of $\mathbf{M}\mu$. In what follows, we write $\vdash$ for $\vdash_{\mathbf{M}\mu}$.

4 Completeness

Given a finite set Γ of $\mathbf{M}\mu$ sequents, we construct a model M_Γ such that for all $\mathbf{M}\mu$ sequents $\mathcal{I}$, $M_\Gamma \models \mathcal{I}$ if and only if $\Gamma \vdash \mathcal{I}$.

Definition For any finite set Γ of $\mathbf{M}\mu$ sequents, M_Γ is the model such that

- $|M_\Gamma|$ consists of all finite sequences X of atomic formulas and all expressions of the form $X \mid Y$, where X is a non-empty finite sequence of atomic formulas and Y is any finite sequence of atomic formulas.
- For $\alpha, \beta \in |M_\Gamma|$, $\langle \alpha, \beta \rangle \in [\![p]\!]_{M_\Gamma}$ if and only if one of the following holds:
  (i) $\beta = \alpha\, p$
  (ii) $\alpha = X$ and $\beta = X\, p \mid$ for some X.
  (iii) $\alpha = X\, Y$, $\beta = X$, and $\Gamma \vdash X \Rightarrow Y\, p$ for some X, Y.
  (iv) $\alpha = X \mid Y\, Z$, $\beta = X \mid Y$, and $\Gamma \vdash X\, Y \Rightarrow Z\, p$ for some X, Y, Z.

In what follows, α, β, γ (with or without subscripts) range over elements of $|M_\Gamma|$. We use $\mathbf{\Lambda}$ to denote the empty expression (which is in $|M_\Gamma|$). For every $\alpha \in |M_\Gamma|$, let $(\alpha)^\dagger = X\, Y$ if $\alpha = X \mid Y$, and $(\alpha)^\dagger = X$ if $\alpha = X$. Let $\mathrm{lh}(\alpha)$ be the number of occurrences of atomic formulas in $(\alpha)^\dagger$.

Definition An expression of the form

$$\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$$

is called a *path* (from α_0 to α_n) (in M_Γ) if $\langle \alpha_0, \alpha_1 \rangle \in [\![d_1]\!]_{M_\Gamma}, \ldots, \langle \alpha_{n-1}, \alpha_n \rangle \in [\![d_n]\!]_{M_\Gamma}$. The *label* of this path is $d_1 \ldots d_n$.

The bar $\mid$ in $X \mid Y \in |M_\Gamma|$ is there to indicate that there is no way to get back from $X \mid Y$ to an initial segment X' of X. If α_0 $(= X \mid Y)$ $\xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is a path, $X \mid$ is an initial segment of each α_i $(0 \le i \le n)$.

Definition A path $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is called *non-shrinking* if $\mathrm{lh}(\alpha_0) \le \mathrm{lh}(\alpha_i)$ for $0 \le i \le n$. A non-shrinking path $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is called a *loop* if $n \ge 1$ and $\alpha_0 = \alpha_n$. A *minimal loop* is a loop with no proper subloop (i.e., a loop such that no proper subpath of it is a loop).

If $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is a non-shrinking path, α_0 is an initial segment of each α_i $(0 \le i \le n)$. The following lemma is straightforward.

Lemma 4.1 *Let α and β be elements of $|M_\Gamma|$ such that $\beta = \alpha\gamma$ for some $\gamma \in |M_\Gamma|$. Then there exists a unique shortest path from α to β, and the label of this path is $(\gamma)^\dagger$.*

Lemma 4.2 *A non-shrinking path from α to β that does not contain a loop is the shortest path from α to β.*

PROOF. Let

$$\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n \tag{1}$$

be a non-shrinking path. (1) is the shortest path from α_0 to α_n if and only if $\mathrm{lh}(\alpha_{i+1}) = \mathrm{lh}(\alpha_i) + 1$ for $0 \le i < n$. Suppose that (1) is not the shortest path from α_0 to α_n, so that for some i such that $0 \le i < n$, $\mathrm{lh}(\alpha_i) \ge \mathrm{lh}(\alpha_{i+1})$. Pick the smallest such i. Then (1) must begin with

$$\alpha_0\ (= \alpha\,\gamma_0) \xrightarrow{d_1} \alpha_0\,\gamma_1 \xrightarrow{d_2} \cdots \xrightarrow{d_i} \alpha_0\,\gamma_i$$

where $(\gamma_k)^\dagger = d_1 \ldots d_k$ $(0 \le k \le i)$, and α_{i+1} must be $\alpha_0\,\gamma_j$ for some j such that $0 \le j \le i$. Thus,

$$\alpha_0\,\gamma_j \xrightarrow{d_{j+1}} \cdots \xrightarrow{d_i} \alpha_0\,\gamma_i \xrightarrow{d_{i+1}} \alpha_0\,\gamma_j$$

is a loop, which is contained in (1). ∎

Lemma 4.3 *If $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is a minimal loop, then $\Gamma \vdash (\alpha_0)^\dagger \Rightarrow d_1 \ldots d_n$.*

PROOF. The path $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \alpha_{n-1}$ is non-shrinking and does not contain a loop, so by Lemma 4.2, it is the shortest path from α_0 to α_{n-1}. Then it must be that $\alpha_{n-1} = \alpha_0\, d_1 \ldots d_{n-1}$, and the lemma follows. ∎

Lemma 4.4 *If $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is a loop, then $\Gamma \vdash (\alpha_0)^\dagger \Rightarrow d_1 \ldots d_n$.*

PROOF. By induction on the number k of proper subloops in $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$.

CASE 1. $k = 0$. By Lemma 4.3.

CASE 2. $k \ge 1$. Take the leftmost minimal subloop $\alpha_i \xrightarrow{d_{i+1}} \cdots \xrightarrow{d_j} \alpha_j\ (= \alpha_i)$ of $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$. The path $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_i} \alpha_i$ is non-shrinking and does not contain a loop, so by Lemma 4.2, it is the shortest path from α_0 to α_i. Note that $\alpha_i = \alpha_0\gamma$ for some γ. Then, by Lemma 4.1, $d_1 \ldots d_i = (\gamma)^\dagger$, so $(\alpha_i)^\dagger = (\alpha_0)^\dagger d_1 \ldots d_i$. Hence by Lemma 4.3,

$$\Gamma \vdash (\alpha_0)^\dagger\, d_1 \ldots d_i \Rightarrow d_{i+1} \ldots d_j \tag{2}$$

Moreover, $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_i} \alpha_i\ (= \alpha_j) \xrightarrow{d_{j+1}} \cdots \xrightarrow{d_n} \alpha_n$ is a loop with no more than $k-1$ subloops, so by the induction hypothesis,

$$\Gamma \vdash (\alpha_0)^\dagger \Rightarrow d_1 \ldots d_i\, d_{j+1} \ldots d_n \tag{3}$$

From (2) and (3), we get

$$\Gamma \vdash (\alpha_0)^\dagger \Rightarrow d_1 \ldots d_i\, d_{i+1} \ldots d_j\, d_{j+1} \ldots d_n$$

by $(\Rightarrow \bullet)$. ∎

Corollary 4.5 *If*

$$\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$$

is a path and $\alpha_0 = \alpha_n = X \mid$ *for some* X*, then*

$$\Gamma \vdash X \Rightarrow d_1 \ldots d_n.$$

PROOF. Every path that starts from $X \mid$ must be non-shrinking, so $\alpha_0 \xrightarrow{d_1} \cdots \xrightarrow{d_n} \alpha_n$ is a loop. ∎

Lemma 4.6 *If* $\Gamma \vdash \mathcal{I}$*, then* $M_\Gamma \models \mathcal{I}$*.*

PROOF. Assume that $\Gamma \vdash p_1 \ldots p_n \Rightarrow d_1 \ldots d_m$ and there is a path

$$\alpha_0 \xrightarrow{p_1} \cdots \xrightarrow{p_n} \alpha_n.$$

Given the construction of M_Γ, it suffices to show

$$\Gamma \vdash (\alpha_n)^\dagger \Rightarrow d_1 \ldots d_m \tag{4}$$

for, then, there is a loop

$$\alpha_n \xrightarrow{d_1} \alpha_n\, d_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{m-1}} \alpha_n\, d_1 \ldots d_{m-1} \xrightarrow{d_m} \alpha_n.$$

Since $\Gamma \vdash p_1 \ldots p_n \Rightarrow d_1 \ldots d_m$ by assumption, $\Gamma \vdash (\alpha_0)^\dagger\, p_1 \ldots p_n \Rightarrow d_1 \ldots d_m$ by Left Monotonicity. Note that, by Lemma 4.1, $(\alpha_0)^\dagger$ is the label of the shortest path from $\mathbf{\Lambda}$ to α_0, and so $(\alpha_0)^\dagger\, p_1 \ldots p_n$ is the label of a path from $\mathbf{\Lambda}$ to α_n. Thus, to prove (4), it suffices to show that if there is a path

$$\beta_0\,(= \mathbf{\Lambda}) \xrightarrow{c_1} \cdots \xrightarrow{c_l} \beta_l \tag{5}$$

and

$$\Gamma \vdash c_1 \ldots c_l \Rightarrow d_1 \ldots d_m \tag{6}$$

then

$$\Gamma \vdash (\beta_l)^\dagger \Rightarrow d_1 \ldots d_m \tag{7}$$

We prove this by induction on the number k of loops in (5).

CASE 1. $k = 0$. Then, by Lemma 4.2, (5) is the shortest path from $\mathbf{\Lambda}$ to β_l and, by Lemma 4.1, $c_1 \ldots c_l = (\beta_l)^\dagger$. So (6) is (7).

CASE 2. $k \geq 1$. Take a leftmost loop

$$\beta_i \xrightarrow{c_{i+1}} \cdots \xrightarrow{c_j} \beta_j\,(= \beta_i)$$

in (5). Since by Lemma 4.2 $\beta_0 \xrightarrow{c_1} \cdots \xrightarrow{c_i} \beta_i$ must be the shortest path from $\beta_0\,(= \mathbf{\Lambda})$ to β_i, $c_1 \ldots c_i = (\beta_i)^\dagger$ by Lemma 4.1. By Lemma 4.4, then,

$$\Gamma \vdash c_1 \ldots c_i \Rightarrow c_{i+1} \ldots c_j \tag{8}$$

From (6) and (8),

$$\Gamma \vdash c_1 \ldots c_i\, c_{j+1} \ldots c_l \Rightarrow d_1 \ldots d_m$$

by Left Cut. But

$$\beta_0\,(= \mathbf{\Lambda}) \xrightarrow{c_1} \cdots \xrightarrow{c_i} \beta_i\,(= \beta_j) \xrightarrow{c_{j+1}} \cdots \xrightarrow{c_l} \beta_l$$

is a path with no more than $k - 1$ loops, so the induction hypothesis applies to give (7). ∎

Theorem 1 *$M_\Gamma \models \mathcal{I}$ if and only if $\Gamma \vdash \mathcal{I}$.*

Proof. The *if* direction is Lemma 4.6, and the *only if* direction follows from Corollary 4.5, noting that $p_1 \dots p_n \mid\in \mathrm{range}([\![p_1]\!]_{M_\Gamma} \circ \cdots \circ [\![p_n]\!]_{M_\Gamma})$, and the fact that every path starting from $\boldsymbol{\Lambda}$ must be non-shrinking (to take care of the case of empty antecedent). ■

5 Filtration

That $\mathbf{M}\mu$ has the finite model property can be shown by the method of filtration.

Let a finite set Γ of $\mathbf{M}\mu$ sequents and an $\mathbf{M}\mu$ sequent $p_1 \dots p_N \Rightarrow d_1 \dots d_L$ be given, and suppose that $M \models \Gamma$ and $M \not\models p_1 \dots p_N \Rightarrow d_1 \dots d_L$. Below, we shall describe a method of constructing a finite model M_0 such that $M_0 \models \Gamma$ and $M_0 \not\models p_1 \dots p_N \Rightarrow d_1 \dots d_L$.

In this section, we write

$$x \xrightarrow{q} y$$

to mean $\langle x, y\rangle \in [\![q]\!]_M$. Let $\mathcal{P}$ be the (finite) set of atomic formulas that appear in $\Gamma \cup \{p_1 \dots p_N \Rightarrow d_1 \dots d_L\}$.

Definition Let $\boldsymbol{w} \in |M|$ be such that $\boldsymbol{w} \in \mathrm{range}([\![p_1]\!]_M \circ \cdots \circ [\![p_N]\!])$ and $\boldsymbol{w} \notin \mathrm{fix}([\![d_1]\!]_M \circ \cdots \circ [\![d_L]\!]_M)$. For each natural number n, define an equivalence relation $\equiv_n$ on $|M|$ by induction as follows. For every $x, y \in |M|$,

$$\begin{array}{rll} x \equiv_0 y & \text{iff} & x = y = \boldsymbol{w} \text{ or } x \neq \boldsymbol{w}, y \neq \boldsymbol{w} \\ x \equiv_{n+1} y & \text{iff} & \text{(i) } x \equiv_n y \text{ and} \\ & & \text{(ii) for all } z \in |M| \text{ and all } q \in \mathcal{P}, \\ & & \quad \text{if } z \xrightarrow{q} x \text{, then for some } v \in |M|, \\ & & \quad\quad v \xrightarrow{q} y \text{ and } z \equiv_n v \\ & & \quad \text{and vice versa.} \end{array}$$

Lemma 5.1 *For each n, $\equiv_n$ has only finitely many equivalence classes.*

Proof. Induction on n. The basis $n = 0$ is obvious. For the induction step, assume that $\equiv_n$ has $f(n)$ equivalence classes. The clause (ii) of the definition of $\equiv_{n+1}$ can be rewritten as

$$\{\, \langle q, [z]_{\equiv_n}\rangle \mid q \in \mathcal{P}, z \in |M|, z \xrightarrow{q} x \,\} = \{\, \langle q, [z]_{\equiv_n}\rangle \mid q \in \mathcal{P}, z \in |M|, z \xrightarrow{q} y \,\}.$$

($[z]_{\equiv_n} = \{\, v \in |M| \mid z \equiv_n v \,\}$.) This makes it clear that $\equiv_{n+1}$ has at most $f(n) \cdot 2^{\mathrm{card}(\mathcal{P}) \cdot f(n)}$ equivalence classes. ($\mathrm{card}(A)$ is the cardinality of A.) ■

Lemma 5.2 *Let $k \leq n$. If*

$$x_0 \equiv_n y_0 \xrightarrow{c_1} x_1 \equiv_n y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} x_k \equiv_n y_k,$$

then there are $z_0, \dots, z_k$ such that

$$x_0 \equiv_{n-k} z_0 \xrightarrow{c_1} z_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} z_k \equiv_n y_k.$$

Proof. Induction on k. The case $k = 0$ is obvious. Let $k \geq 1$ and suppose

$$x_0 \equiv_n y_0 \xrightarrow{c_1} x_1 \equiv_n y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} x_k \equiv_n y_k.$$

By induction hypothesis, there are $z_1, \ldots, z_k$ such that

$$x_1 \equiv_{n-k+1} z_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} z_k \equiv_n y_k.$$

Since $n - k + 1 \geq 1$ and $y_0 \xrightarrow{c_1} x_1$ and $x_1 \equiv_{n-k+1} z_1$, there must be a z_0 such that $z_0 \xrightarrow{c_1} z_1$ and $y_0 \equiv_{n-k} z_0$. Since $x_0 \equiv_n y_0$, we have $x_0 \equiv_{n-k} z_0$. ∎

Corollary 5.3 *Let $k \leq n$. If*

$$\boldsymbol{w} = x_0 \equiv_n y_0 \xrightarrow{c_1} x_1 \equiv_n y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} x_k \equiv_n y_k = \boldsymbol{w},$$

then there are $z_0, \ldots, z_k$ such that

$$\boldsymbol{w} = z_0 \xrightarrow{c_1} z_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} z_k = \boldsymbol{w}.$$

Definition For each n, define a model $M/{\equiv_n}$ as follows:

- $|M/{\equiv_n}| = \{\, [x]_{\equiv_n} \mid x \in |M| \,\}$.
- $[\![q]\!]_{M/\equiv_n} = \{\, \langle [x]_{\equiv_n}, [y]_{\equiv_n} \rangle \mid \langle x, y \rangle \in [\![q]\!]_M \,\}$.

By Lemma 5.1, $M/{\equiv_n}$ is a finite model for each n.

Let K be the maximal length of antecedents in Γ.

Lemma 5.4 *If $n \geq K$, $M/{\equiv_n} \models \Gamma$.*

Proof. Let $c_1 \ldots c_k \Rightarrow q_1 \ldots q_j \in \Gamma$, and let $[y_k]_{\equiv_n} \in \mathrm{range}([\![c_1]\!]_{M/\equiv_n} \circ \cdots \circ [\![c_k]\!]_{M/\equiv_n})$. Then there must be $y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k$ such that

$$y_0 \xrightarrow{c_1} x_1 \equiv_n y_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} x_k \equiv_n y_k.$$

$k \leq K \leq n$. Then, by Lemma 5.2, there are $z_0, \ldots, z_k$ such that

$$z_0 \xrightarrow{c_1} \cdots \xrightarrow{c_k} z_k \equiv_n y_k.$$

Since $M \models \Gamma$, there must be $v_0, \ldots, v_j$ such that

$$z_k = v_0 \xrightarrow{q_1} \cdots \xrightarrow{q_j} v_j = z_k.$$

It follows that $[y_k]_{\equiv_n} \in \mathrm{fix}([\![q_1]\!]_{M/\equiv_n} \circ \cdots \circ [\![q_j]\!]_{M/\equiv_n})$. ∎

Lemma 5.5 *If $n \geq L$, $[\boldsymbol{w}]_{\equiv_n} \notin \mathrm{fix}([\![d_1]\!]_{M/\equiv_n} \circ \cdots \circ [\![d_L]\!]_{M/\equiv_n})$.*

Proof. Immediate from Corollary 5.3 and the assumption about $\boldsymbol{w}$. ∎

Theorem 2 *If $n \geq \max(K, L)$, then $M/{\equiv_n}$ is a finite model such that $M/{\equiv_n} \models \Gamma$ and $M/{\equiv_n} \not\models p_1 \ldots p_N \Rightarrow d_1 \ldots d_L$.*

Proof. By Lemmas 5.1, 5.4, 5.5, and the fact that $[\boldsymbol{w}]_{\equiv_n} \in \mathrm{range}([\![p_1]\!]_{M/\equiv_n} \circ \cdots \circ [\![p_N]\!]_{M/\equiv_n})$. ∎

6 Reduction to Propositional Dynamic Logic with Intersection

Decidability of $\mathbf{M}\mu$ can also be shown by a translation into propositional dynamic logic with intersection, which is known to be decidable (Danecki 1985). In what follows, I assume familiarity with propositional dynamic logic.

Definition Let $\mathrm{tr}(p_1 \dots p_n \Rightarrow d_1 \dots d_m) = [p_1] \cdots [p_n]\langle (d_1; \cdots; d_m) \cap (\top)? \rangle \top$. Let $\mathrm{tr}(\{\mathcal{I}_1, \dots, \mathcal{I}_k\}) = \mathrm{tr}(\mathcal{I}_1) \wedge \cdots \wedge \mathrm{tr}(\mathcal{I}_k)$.

Lemma 6.1 *Let $\mathcal{I}$ be an $\mathbf{M}\mu$ sequent. $M \models \mathcal{I}$ in mixed inference if and only if $M \models \mathrm{tr}(\mathcal{I})$ in propositional dynamic logic with intersection.*

Lemma 6.2 *Let Γ be a finite set of $\mathbf{M}\mu$ sequents, $\mathcal{I}$ be an $\mathbf{M}\mu$ sequent, and $q_1, \dots, q_l$ be the atomic formulas occurring in $\Gamma \cup \{\mathcal{I}\}$. Then $\Gamma \models \mathcal{I}$ in mixed inference if and only if $\models [(q_1 \cup \cdots \cup q_l)^*]\mathrm{tr}(\Gamma) \to \mathrm{tr}(\mathcal{I})$ in propositional dynamic logic with intersection.*

Proof. The *if* direction is clear. For the *only if* direction, assume that $M, w \models [(q_1 \cup \cdots \cup q_l)^*]\mathrm{tr}(\Gamma)$ and $M, w \models \neg\mathrm{tr}(\mathcal{I})$. Let M_0 be the submodel of M whose states are those that can be reached from w via $(q_1 \cup \cdots \cup q_l)^*$. Then $M_0 \models \mathrm{tr}(\Gamma)$ and $M_0, w \models \neg\mathrm{tr}(\mathcal{I})$. This means that $M_0 \models \Gamma$ and $M_0 \not\models \mathcal{I}$, so $\Gamma \not\models \mathcal{I}$. ■

7 Deterministic Models

Let us call a model $M = \langle |M|, R_0, R_1, R_2, \dots \rangle$ where each R_i is a partial function a *deterministic model*. It is interesting to consider mixed inference with respect to the class of deterministic models, because of the close connection with update semantics of Veltman (1991).[3]

In the simple case where there is no connective and the succedent of the sequent is a single formula, addition of the following rule to $\mathbf{M}$ results in a calculus complete with respect to deterministic models. Let us call the resulting calculus $\mathbf{U}$.

Cautious Monotonicity $\dfrac{X \Rightarrow c \quad X\,Y \Rightarrow d}{X\,c\,Y \Rightarrow d}$

In the multiple succedent case (which is equivalent to having composition), more rules become necessary:

Cautious Monotonicity $\dfrac{X \Rightarrow Y \quad X\,Z \Rightarrow W}{X\,Y\,Z \Rightarrow W}$

$(\Rightarrow \bullet^{-1})$ $\dfrac{X\,Y \Rightarrow Z \quad X \Rightarrow Y\,Z\,W}{X \Rightarrow Y\,W}$

Rotation $\dfrac{X \Rightarrow Y\,Z}{X\,Y \Rightarrow Z\,Y}$

The calculus which results from adding the above three rules to $\mathbf{M}\mu$ is called $\mathbf{U}\mu$. $\mathbf{U}\mu$ can be shown to be complete with respect to deterministic models. I state the necessary results without proof.

Definition For any finite set Γ of multiple-succedent sequents, $M_\Gamma^{\mathbf{U}\mu}$ is the model such that

[3] A difference between update semantics and mixed inference with respect to deterministic models is that Reflexivity $P \Rightarrow P$ holds in the former, but not in the latter.

– $|M_\Gamma^{\mathbf{U}\mu}| = \{\, X \mid X$ is a finite sequence of atomic formulas and $\neg\exists X_1 X_2 X_3 (X = X_1\, X_2\, X_3 \wedge X_2 \neq \mathbf{\Lambda} \wedge \Gamma \vdash_{\mathbf{U}\mu} X_1 \Rightarrow X_2)\,\}$.

– $[\![p]\!]_{M_\Gamma^{\mathbf{U}\mu}} = \{\, \langle X, X\, p\rangle \mid X, X\, p \in |M_\Gamma^{\mathbf{U}\mu}|\,\} \cup \{\, \langle X\, Y, X\rangle \mid X\, Y \in |M_\Gamma^{\mathbf{U}\mu}|, \Gamma \vdash X \Rightarrow Y\, p\,\}$.

Lemma 7.1 *In $M_\Gamma^{\mathbf{U}\mu}$, the interpretation of each atomic formula is a total function.*

Below, I write

$$X \xrightarrow{d_1 \dots d_n} Y$$

to mean $\langle X, Y\rangle \in [\![d_1]\!]_{M_\Gamma^{\mathbf{U}\mu}} \circ \cdots \circ [\![d_n]\!]_{M_\Gamma^{\mathbf{U}\mu}}$.

Lemma 7.2 *If $\mathbf{\Lambda} \xrightarrow{W} X$, then $\Gamma \cup \{X \Rightarrow Y\} \vdash_{\mathbf{U}\mu} W \Rightarrow Y$ for any Y.*

Lemma 7.3 *If $X \xrightarrow{Y} X$, then $\Gamma \vdash_{\mathbf{U}\mu} X \Rightarrow Y$.*

Lemma 7.4 *If $M_\Gamma^{\mathbf{U}\mu} \models X \Rightarrow Y$, then $\Gamma \vdash_{\mathbf{U}\mu} X \Rightarrow Y$.*

Lemma 7.5 *If $\mathbf{\Lambda} \xrightarrow{W} X$, then $\Gamma \cup \{W \Rightarrow Y\} \vdash_{\mathbf{U}\mu} X \Rightarrow Y$ for any Y.*

Lemma 7.6 *If $\Gamma \vdash_{\mathbf{U}\mu} X \Rightarrow Y$ and $X \in |M_\Gamma^{\mathbf{U}\mu}|$, then $X \xrightarrow{Y} X$.*

Lemma 7.7 *If $\Gamma \vdash_{\mathbf{U}\mu} X \Rightarrow Y$, then $M_\Gamma^{\mathbf{U}\mu} \models X \Rightarrow Y$.*

Theorem 3 *$\Gamma \vdash_{\mathbf{U}\mu} X \Rightarrow Y$ if and only if $M_\Gamma^{\mathbf{U}\mu} \models X \Rightarrow Y$.*

Theorem 3 shows that $\mathbf{U}\mu$ is complete with respect to models where the interpretation of each atomic formula is a *total* function.

The calculus $\mathbf{U}$ for the single succedent case is decidable. As in the case of $\mathbf{M}$, this is easy to see by translation into first-order logic, noting that the partial functionality of the relevant relations can be expressed by universal first-order sentences. On the other hand, I have been unable to prove the decidability of $\mathbf{U}\mu$. The method of filtration used in Section 5 does not necessarily lead to deterministic models, so proof of the finite model property would require additional work.[4]

8 Other Styles of Inference

Let us consider the styles of inference determined by the following stipulations:

(a) $M \models P_1 \dots P_n \Rightarrow C$ if and only if $[\![P_1]\!]_M \circ \cdots \circ [\![P_n]\!]_M \subseteq [\![C]\!]_M$.

(b) $M \models P_1 \dots P_n \Rightarrow C$ if and only if $\mathrm{range}([\![P_1]\!]_M \circ \cdots \circ [\![P_n]\!]_M) \subseteq \mathrm{dom}([\![C]\!]_M)$.

(c) $M \models P_1 \dots P_n \Rightarrow C$ if and only if $\mathrm{dom}([\![P_1]\!]_M \circ \cdots \circ [\![P_n]\!]_M) \subseteq \mathrm{dom}([\![C]\!]_M)$.

The first notion (a), called *dynamic inference* in van Benthem 1991, is axiomatized as follows. The first calculus $\mathbf{L}$ is for the single succedent (connective-free) case, and the second calculus $\mathbf{L}\mu$ is for the multiple succedent case.

[4]Note also that the problem of whether or not a formula of propositional dynamic logic with intersection has a deterministic model is Σ_1^1-hard (Harel 1983).

Calculus L.

– Axiom: Reflexivity $p \Rightarrow p$

– Rule of Inference:

$$\text{Cut} \quad \frac{X \Rightarrow c \quad Y\, c\, Z \Rightarrow d}{Y\, X\, Z \Rightarrow d}$$

Calculus Lμ.

– Axiom: Reflexivity $X \Rightarrow X$

– Rule of Inference:

$$\text{Cut} \quad \frac{X \Rightarrow Y \quad Z\, Y\, W \Rightarrow V}{Z\, X\, W \Rightarrow V}$$

One can extract from **L**μ the calculus **L**$(\bullet)$ for dynamic inference with composition. **L** and **L**$(\bullet)$ are fragments of the Lambek calculus, and the completeness of **L** and **L**$(\bullet)$ (or **L**μ) is a consequence of the known strong completeness of the Lambek calculus with respect to relational semantics (Mikulás 1992).

The problem '$\Gamma \vdash_{\mathbf{L}} X \Rightarrow d$?' is decidable in $O(n^5)$ time, while the problem '$\Gamma \vdash_{\mathbf{L}\mu} X \Rightarrow Y$?' (or, equivalently, '$\Gamma \vdash_{\mathbf{L}(\bullet)} X \Rightarrow D$?') is undecidable. This follows from the observation that the first problem is equivalent to the universal membership problem for context-free grammars, and the second to that for semi-Thue systems (Type 0 grammars). That is, if we reverse the arrows of sequents, single-succedent sequents behave just like rules of context-free grammars, and multiple-succedent sequents behave just like unrestricted rewriting rules. Reflexivity and Cut have the effect of taking the reflexive transitive closure of one-step rewriting, and derivations in the two calculi precisely correspond to the derivations in the respective types of grammars. The undecidability of **L**μ (or, equivalently, of **L**$(\bullet)$) contrasts with the situation with mixed inference and the other two styles of inference considered below.

The remaining two styles of inference, (b) and (c), are axiomatized by the following calculi **G**μ and **E**μ, respectively (in the multiple succedent case).[5]

Calculus Gμ.

– Axioms: **G**μ has no axioms.

– Rules of Inference:

$$\text{Left Monotonicity} \quad \frac{X \Rightarrow Y}{p\, X \Rightarrow Y}$$

$$\text{Right Anti-Monotonicity} \quad \frac{X \Rightarrow Y\, d}{X \Rightarrow Y}$$

$$(\Rightarrow \bullet_3) \quad \frac{X\, Y \Rightarrow Z \quad X \Rightarrow Y}{X \Rightarrow Y\, Z}$$

Calculus Eμ.

– Axiom: Reflexivity $X \Rightarrow X$

[5]The style of inference given by (b) is related to dynamic predicate logic of Groenendijk and Stokhof (1991).

– Rules of Inference:

Right Monotonicity $\dfrac{X \Rightarrow Y}{X\, p \Rightarrow Y}$

Right Cut $\dfrac{X \Rightarrow Y \quad Z\, Y \Rightarrow W}{Z\, X \Rightarrow W}$

The completeness of $\mathbf{G}\mu$ and of $\mathbf{E}\mu$ can be shown by a canonical model construction. Here, I only note the definitions of canonical models.

Definition For any finite set Γ of multiple-succedent sequents, $M_\Gamma^{\mathbf{G}\mu}$ is the model such that

- $|M_\Gamma^{\mathbf{G}\mu}|$ consists of all finite sequences X of atomic formulas and all expressions of the form $X \mid Y$ where X and Y are finite sequences of atomic formulas.
- For $\alpha, \beta \in |M_\Gamma^{\mathbf{G}\mu}|$, $\langle \alpha, \beta \rangle \in [\![p]\!]_{M_\Gamma^{\mathbf{G}\mu}}$ if and only if one of the following holds:

  (i) $\alpha = X$ and $\beta = X\, p$ for some X.

  (ii) $\alpha = X$ and $\beta = X\, p \mid$ for some X.

  (iii) $\alpha = X \mid Y$, $\beta = X \mid Y\, p$, and $\Gamma \vdash_{\mathbf{G}\mu} X \Rightarrow Y\, p$ for some X, Y.

Definition For any finite set Γ of multiple-succedent sequents, $M_\Gamma^{\mathbf{E}\mu}$ is the model such that

- $|M_\Gamma^{\mathbf{E}\mu}|$ consists of all expressions of the form $X \mid Y$ where X and Y are finite sequences of atomic formulas.
- For $\alpha, \beta \in |M_\Gamma^{\mathbf{E}\mu}|$, $\langle \alpha, \beta \rangle \in [\![p]\!]_{M_\Gamma^{\mathbf{E}\mu}}$ if and only if $\alpha = X \mid Y$, $\beta = X \mid Y\, p$, and $\Gamma \vdash_{\mathbf{E}\mu} X \Rightarrow Y\, p$ for some X, Y.

The finite model property of $\mathbf{G}\mu$ can be proved in exactly the same way as for $\mathbf{M}\mu$, using the same definition of $\equiv_n$. As for $\mathbf{E}\mu$, a minor modification (using the 'forward' version of $\equiv_n$) works. Also, the decidability of $\mathbf{G}\mu$ and $\mathbf{E}\mu$ can be shown by reduction to propositional dynamic logic (this time using only regular program constructions).

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