Computing Interpolants in Implicational Logics

Makoto Kanazawa

National Institute of Informatics
2–1–2 Hitotsubashi, Chiyoda-ku, Tokyo, 101–8430, Japan

Abstract

I present a new syntactical method for proving the Interpolation Theorem for the implicational fragment of intuitionistic logic and its substructural subsystems. This method, like Prawitz's, works on natural deductions rather than sequent derivations, and, unlike existing methods, always finds a 'strongest' interpolant under a certain restricted but reasonable notion of what counts as an 'interpolant'.

Keywords: interpolation, natural deduction, simply typed λ-calculus, intuitionistic logic

1 Introduction

This work is motivated by the following problem in the simply typed λ-calculus:

Problem. Given a normal term $T[\vec{x}, \vec{y}]$, find two normal terms $S[\vec{x}]$ and $P[z, \vec{y}]$ such that $P[S[\vec{x}], \vec{y}] \twoheadrightarrow_\beta T[\vec{x}, \vec{y}]$.

The question of how to solve this problem occupies a central place in a certain computational model of acquisition of word meanings by children (Kanazawa 2001, 2003).[1] Finding one solution to this problem is easy; any instance of the problem has the following solution:

$$S[\vec{x}] = \lambda w.w\vec{x},$$
$$P[z, \vec{y}] = z(\lambda \vec{x}.T[\vec{x}, \vec{y}]).$$

However, there are many other solutions, and one particularly interesting class of solutions, from the standpoint of the computational model mentioned above, consists of those solutions that assign a 'simplest' type to S.

It turns out that standard syntactical proofs of the *Interpolation Theorem* for intuitionistic logic provide algorithms for finding such solutions. There are two well-known syntactical methods for proving the Interpolation Theorem, one by Maehara (1960) (see Troelstra and Schwichtenberg 2000 for the history and details of the method) and one by Prawitz (1965). Maehara's method works by induction

[1] We cannot not go into any details in this paper, but very briefly, the model assumes that meanings of words and sentences, as well as ways of combining word meanings to build sentence meanings (called "meaning recipes"), are represented by typed λ-terms. Meanings of words and sentences contain constants that represent "semantic primitives", but meaning recipes are pure λ-terms without constants. Suppose that a child encounters a sentence whose meaning $T[\vec{c}, \vec{d}]$ (with constants $\vec{c}, \vec{d}$) is clear to her but which contains one word new to her. If she can tell that constants $\vec{c}$ come from the unknown word and $\vec{d}$ come from the rest of the sentence, then finding out the meaning of the unknown word consists in finding an appropriate pair of terms $S[\vec{x}]$, $P[z, \vec{y}]$ such that $P[S[\vec{x}], \vec{y}] \twoheadrightarrow_\beta T[\vec{x}, \vec{y}]$.

on cut-free sequent derivations, and Prawitz's method works by induction on normal natural deductions. In these methods, what is to be proved by induction is the following statement:

Interpolation Theorem. *If* $\vdash \Gamma, \Delta \Rightarrow C$*, then there is a formula* E *such that*

- $\vdash \Gamma \Rightarrow E$*;*
- $\vdash E, \Delta \Rightarrow C$*;*
- *all propositional variables in* E *appear both in* Γ *and in* Δ, C.

A formula E satisfying the above conditions is called an *interpolation formula* to the sequent $\Gamma, \Delta \Rightarrow C$ with respect to the *partition* $(\Gamma; \Delta)$ of its antecedent. Implicit in the inductive proof of this statement is an algorithm that, given a cut-free derivation/normal deduction $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$, finds two cut-free derivations/normal deductions $\mathscr{D}_1\colon \Gamma \Rightarrow E$ and $\mathscr{D}_0\colon E, \Delta \Rightarrow C$. Crucially, the two derivations/deductions $\mathscr{D}_1$ and $\mathscr{D}_0$ found by these methods in fact satisfy much stronger properties. Assuming that $\Gamma, \Delta \Rightarrow C$ consists of implicational formulas, let $T[\vec{x}, \vec{y}], S[\vec{x}], P[z, \vec{y}]$ be the λ-terms corresponding to $\mathscr{D}, \mathscr{D}_1, \mathscr{D}_0$, respectively, where the types of the variables $\vec{x}$ are the formulas in Γ, the types of the variables $\vec{y}$ are the formulas in Δ, and the type of z is E. Then one has:

(i) $P[S[\vec{x}], \vec{y}] \twoheadrightarrow_\beta T[\vec{x}, \vec{y}]$;

(ii) Both in $\mathscr{D}_1\colon \Gamma \Rightarrow E$ and in $\mathscr{D}_0\colon E, \Delta \Rightarrow C$, no occurrence of a propositional variable inside E is *linked* to another such occurrence or originates in an application of Weakening.

Condition (ii) is stated in terms of sequent calculus. In a cut-free sequent derivation, two occurrences of a propositional variable in the endsequent are *linked* to each other if they originate 'opposite to' each other in an initial sequent. The condition is invariant across cut-free derivations corresponding to the same normal natural deduction that are *W-normal* in the sense of Mints (1996), and it can be stated directly in terms of natural deduction as well. So (ii) is a property of the λ-terms S, P. Condition (i) is emphasized by Čubrić (1994) for Prawitz's method, and condition (ii) is a strengthening of one of the conditions stated by Carbone (1997) in terms of sequent calculus.

Deviating from standard terminology, we say that a normal term $S[\vec{x}]$ is an *interpolant* to a normal term $T[\vec{x}, \vec{y}]$ (with respect to the partition $(\vec{x}; \vec{y})$ of its free variables) if there exists a normal term $P[z, \vec{y}]$ such that S, P satisfy the conditions (i), (ii). The condition (i) simply says that S, P gives a solution to an instance T of our problem. The condition (ii) gives a sense in which E is 'simplest'. It implies that in $\mathscr{D}_1$ and $\mathscr{D}_0$, any occurrence of a propositional variable inside E must be linked to an occurrence outside E, from which the third condition on E in the above statement of the Interpolation Theorem follows.

There are two complications, however. One complication is that the Interpolation Theorem in fact fails to hold in the above form for the implicational fragment of intuitionistic logic, which corresponds to the simply typed λ-calculus. Even when $\Gamma, \Delta \Rightarrow C$ is a sequent consisting entirely of implicational formulas, the interpolation formula E sometimes has to contain conjunction. An example of such a sequent is

$$p_1, p_1 \to p_2, p_1 \to p_3, p_2 \to p_3 \to p_4 \Rightarrow p_4.$$

$p_2 \wedge p_3$ is an interpolation formula to this sequent with respect to the partition $(p_1, p_1 \to p_2, p_1 \to p_3; p_2 \to p_3 \to p_4)$ of its antecedent, but there is no interpolation formula in the implicational fragment.[2]

A way of circumventing this problem has been proposed by Wroński (1984). His idea is to use a sequence of formulas $E_1, \ldots, E_m$ in place of a single formula E in the statement of the Interpolation Theorem. Although Wroński used this idea to prove an Interpolation Theorem for *BCK*-logic, it can readily be extended to the implicational fragment of intuitionistic logic.[3] Thus, we have

Interpolation Theorem. *If $\vdash \Gamma, \Delta \Rightarrow C$, then there is a sequence of formulas $E_1, \ldots, E_m$ such that*

- $\vdash \Gamma \Rightarrow E_i$ *for* $i = 1, \ldots, m$*;*
- $\vdash E_1, \ldots, E_m, \Delta \Rightarrow C$*;*
- *all propositional variables in* $E_1, \ldots, E_m$ *appear both in* Γ *and in* Δ, C.

We call a sequence of formulas $E_1, \ldots, E_m$ satisfying the above conditions an *interpolation sequence* to $\Gamma, \Delta \Rightarrow C$ with respect to the partition $(\Gamma; \Delta)$. Both Maehara's and Prawitz's method can be easily modified to accommodate this change, as we shall see in detail below. In the above example, we can take p_2, p_3 as the desired interpolation sequence.

A second complication is that interpolants in the sense of (i), (ii) (modified to allow sequences of terms $S_1, \ldots, S_m$ in place of S) are by no means unique. In fact, if one applies Maehara's method (in the modified form) to different cut-free sequent derivations corresponding to the λ-term $T[\vec{x}, \vec{y}]$, one may obtain different interpolants. Moreover, there are interpolants that one cannot find by Maehara's method no matter which cut-free derivation corresponding to $T[\vec{x}, \vec{y}]$ one starts with. As for Prawitz's method, it finds one particular interpolant, but there does not seem to be any good way of characterizing this interpolant except to say that it is the one found by Prawitz's method. In particular, both methods sometimes miss interpolants that are 'strongest' in the sense that their types imply the types of all other interpolants.

In section 3.5 of this paper, we give an algorithm for finding a strongest interpolant. This algorithm works by induction on normal natural deductions, but is otherwise quite different from Prawitz's method.

Although we focus on intuitionistic logic, the results in this paper are designed to relativize to substructural subsystems of intuitionistic implicational logic; hence the plural "logics" in the title.[4]

2 Interpolation in Sequent Calculus

In this section, we describe our modification of Maehara's method for the implicational fragment of the sequent calculus *LJ* for intuitionistic logic, as formulated

[2] In relation to this, the condition (ii) must be restated in a somewhat weaker form when the interpolation formula is allowed to contain conjunction. In sequent calculus, the present formulation of (ii) can be maintained by adopting a multiplicative version of $(\wedge\Rightarrow)$ in place of Gentzen's (1935) rules in *LJ*.

[3] Pentus (1997) used the same method to prove interpolation for the product-free Lambek calculus.

[4] See Ono 1998 for information on interpolation for substructural logics.

by Gentzen (1935), and prove that the method satisfies conditions similar to (i) and (ii) in section 1. For this purpose, we use a sequent calculus with λ-term labels, which essentially encode a translation from *LJ* derivations to *NJ* deductions. In this calculus, a sequent is of the form

$$x_1 : A_1, \ldots, x_n : A_n \Rightarrow T : C$$

where $x_1, \ldots, x_n$ are distinct variables, $A_1, \ldots, A_n, C$ are formulas, and T is a term whose free variables are among $x_1, \ldots, x_n$. The antecedent of such a sequent is treated as a set $\{x_1 : A_1, \ldots, x_n : A_n\}$ of variable-labeled formulas. Such a set is called a *context*. We assume that each variable is preassigned a type, so that $x : A$ is short for $x^A : A$, etc. We use $\Gamma, \Delta, \ldots$ to denote contexts. If Γ and Δ are contexts, we write Γ, Δ to denote $\Gamma \cup \Delta$ provided that $\Gamma \cap \Delta = \varnothing$.

***LJ*$_\rightarrow$.**

Initial sequents.

$$x : A \Rightarrow x : A$$

Operational rules for $\rightarrow$.

$$\frac{\Gamma \Rightarrow U : A \quad y : B, \Delta \Rightarrow T : C}{x : A \rightarrow B, \Gamma, \Delta \Rightarrow T[xU/y] : C}\ (\rightarrow\Rightarrow) \qquad \frac{x : A, \Gamma \Rightarrow T : B}{\Gamma \Rightarrow \lambda x.T : A \rightarrow B}\ (\Rightarrow\rightarrow)$$

Structural rules.

$$\frac{y : A, z : A, \Gamma \Rightarrow T : B}{x : A, \Gamma \Rightarrow T[x/y, x/z] : B}\ \text{Contraction} \qquad \frac{\Gamma \Rightarrow T : B}{x : A, \Gamma \Rightarrow T : B}\ \text{Weakening}$$

$$\frac{\Gamma \Rightarrow U : A \quad x : A, \Delta \Rightarrow T : B}{\Gamma, \Delta \Rightarrow T[U/x] : B}\ \text{Cut}$$

In $(\rightarrow\Rightarrow)$, Contraction, and Weakening, x is required to be a fresh variable. By the convention on the use of commas, comma-separated parts of antecedents in these rules must be disjoint. Unlike in Gentzen's original formulation, there is no structural rule of Interchange because antecedents of sequents are treated as sets of variable-labeled formulas.

If $\mathscr{D}$ is a derivation of $\Gamma \Rightarrow T : C$, we write $\mathscr{D} \colon \Gamma \Rightarrow T : C$ to express this fact. The final occurrence of $\Gamma \Rightarrow T : C$ in $\mathscr{D}$ is called the *endsequent* of $\mathscr{D}$.

Cut is eliminable from derivations in $LJ_\rightarrow$ in the sense that whenever one has a derivation $\mathscr{D} \colon \Gamma \Rightarrow T : C$, one can find another derivation $\mathscr{D}' \colon \Gamma \Rightarrow |T|_\beta : C$ which contains no application of Cut, where $|T|_\beta$ is the normal form of T. By leaving out one or both of the structural rules of Contraction and Weakening from $LJ_\rightarrow$, one obtains sequent systems for various *substructural logics*: the relevance logic $R_\rightarrow$, which lacks Weakening; *BCK*-logic, which lacks Contraction; and *BCI*-logic, which lacks both Contraction and Weakening.

We take for granted the notion of *variable renaming*. If $\mathscr{D}$ is a derivation of $\Gamma \Rightarrow T : C$ and σ is a variable renaming, $\mathscr{D}\sigma$ is a derivation of $\Gamma\sigma \Rightarrow T\sigma : C$.

Since λ-term labels appearing in succedents in a derivation can always be recovered from the other information in the derivation, we sometimes omit those labels. We may also occasionally allow ourselves to omit variable labels in the antecedent for the sake of brevity, even though they are not redundant in the same way.

Lemma 1. *The following rules are admissible:*

$$\frac{\Gamma \Rightarrow U : A \quad y : B, \Delta \Rightarrow T : C}{x : A \to B, \Gamma \cup \Delta \Rightarrow T[xU/y] : C}\ (\to\Rightarrow)\dagger \qquad \frac{\Gamma \Rightarrow U : A \quad x : A, \Delta \Rightarrow T : B}{\Gamma \cup \Delta \Rightarrow T[U/x] : B}\ Cut\dagger$$

where it is allowed that $\Gamma \cap \Delta \neq \emptyset$.

Proof. Let

$$\Gamma = \Gamma', x_1 : D_1, \ldots, x_n : D_n,$$
$$\Delta = \Delta', x_1 : D_1, \ldots, x_n : D_n,$$

where $\Gamma' \cap \Delta' = \emptyset$ and let $y_1, \ldots, y_n, z_1, \ldots, z_n$ be fresh variables. Let $\sigma = [y_1/x_1, \ldots, y_n/x_n]$, $\tau = [z_1/x_1, \ldots, z_n/x_n]$, so that $\Gamma\sigma \cap \Delta\tau = \emptyset$. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are derivations of $\Gamma \Rightarrow U : A$ and $y : B, \Delta \Rightarrow T : C$, respectively, we have

$$\cfrac{\cfrac{\begin{array}{c}\mathscr{D}_1\sigma\\ \Gamma', y_1 : D_1, \ldots, y_n : D_n \Rightarrow U\sigma : A\end{array} \quad \begin{array}{c}\mathscr{D}_2\tau\\ y : B, \Delta', z_1 : D_1, \ldots, z_n : D_n \Rightarrow T\tau : C\end{array}}{x : A \to B, \Gamma', y_1 : D_1, \ldots, y_n : D_n, \Delta', z_1 : D_1, \ldots, z_n : D_n \Rightarrow (T\tau)[x(U\sigma)/y] : C}\ (\to\Rightarrow)}{x : A \to B, \Gamma', x_1 : D_1, \ldots, x_n : D_n, \Delta' \Rightarrow T[xU/y] : C}\ \text{Contr}$$

The admissibility of Cut† is proved similarly. □

We adopt the following convention: When we write a derivation in which (→⇒)† or Cut† is used, we mean a derivation in which these rules are eliminated in the way described in the above proof.

2.1 Links in sequent calculus

We associate with each occurrence of a propositional variable in a $LJ_\to$ derivation two *ports*, and call one the *top port* and the other the *bottom port*. We decorate $LJ_\to$ derivations with *links* connecting two ports as follows (p stands for an arbitrary propositional variable):

Initial sequents.

$$x : A[p] \Rightarrow x : A[p]$$

- We draw a link connecting the top port of an occurrence of p in A in the antecedent with the top port of the corresponding occurrence of p in A in the succedent.

Operational rules for →.

(→⇒).

$$\frac{\Gamma[p] \Rightarrow U : A[p] \quad y : B[p], \Delta[p] \Rightarrow T : C[p]}{x : A[p] \to B[p], \Gamma[p], \Delta[p] \Rightarrow T[xU/y] : C[p]}\ (\to\Rightarrow)$$

We draw a link between

- the top port of an occurrence of p in A in the conclusion and the bottom port of the corresponding occurrence of p in A in the left premise;
- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the right premise;

- the top port of an occurrence of p in C in the conclusion and the bottom port of the corresponding occurrence of p in C in the right premise;
- the top port of an occurrence of p in Γ in the conclusion and the bottom port of the corresponding occurrence of p in Γ in the left premise;
- the top port of an occurrence of p in Δ in the conclusion and the bottom port of the corresponding occurrence of p in Δ in the right premise.

($\Rightarrow\rightarrow$).

$$\frac{x : A[p], \Gamma[p] \Rightarrow T : B[p]}{\Gamma[p] \Rightarrow \lambda x.T : A[p] \rightarrow B[p]}\ (\Rightarrow\rightarrow)$$

We draw a link between

- the top port of an occurrence of p in A in the conclusion and the bottom port of the corresponding occurrence of p in A in the premise;
- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the premise;
- the top port of an occurrence of p in Γ in the conclusion and the bottom port of the corresponding occurrence of p in Γ in the premise.

Structural rules.

Contraction.

$$\frac{y : A[p], z : A[p], \Gamma[p] \Rightarrow T : B[p]}{x : A[p], \Gamma[p] \Rightarrow T[x/y, x/z] : B[p]}\ \text{Contraction}$$

We draw a link between

- the top port of an occurrence of p in $x : A$ in the conclusion and the bottom port of the corresponding occurrence of p in $y : A$ in the premise;
- the top port of an occurrence of p in $x : A$ in the conclusion and the bottom port of the corresponding occurrence of p in $z : A$ in the premise;
- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the premise;
- the top port of an occurrence of p in Γ in the conclusion and the bottom port of the corresponding occurrence of p in Γ in the premise.

Weakening.

$$\frac{\Gamma[p] \Rightarrow T : B[p]}{x : A, \Gamma[p] \Rightarrow T : B[p]}\ \text{Weakening}$$

We draw a link between

- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the premise;

- the top port of an occurrence of p in Γ in the conclusion and the bottom port of the corresponding occurrence of p in Γ in the premise.

Cut.

$$\frac{\Gamma[p] \Rightarrow U : A[p] \qquad x : A[p], \Delta[p] \Rightarrow T : B[p]}{\Gamma[p], \Delta[p] \Rightarrow T[U/x] : B[p]}\ \text{Cut}$$

We draw a link between

- the bottom port of an occurrence of p in A in the left premise and the bottom port of the corresponding occurrence of p in A in the right premise;

- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the right premise;

- the top port of an occurrence of p in Γ in the conclusion and the bottom port of the corresponding occurrence of p in Γ in the left premise;

- the top port of an occurrence of p in Δ in the conclusion and the bottom port of the corresponding occurrence of p in Δ in the right premise.

Definition 2. A *path* is a sequence of the form

$$(\rho_1^-, o_1, \rho_1^+, \ldots, \rho_n^-, o_n, \rho_n^+)$$

($n \geq 1$) such that

- for $1 \leq i \leq n$, o_i is an occurrence of a propositional variable and ρ_i^- and ρ_i^+ are distinct ports of o_i;

- for $1 \leq i \leq n-1$, there is a link between ρ_i^+ and ρ_{i+1}^-.

We say that a path $(\rho_1^-, o_1, \rho_1^+, \ldots, \rho_n^-, o_n, \rho_n^+)$ *starts* in o_1 and *ends* in o_n. A path goes through ports of various occurrences of the same propositional variable. Two occurrences of a propositional variable are *linked* to each other if there is a path that starts in one and ends in the other. Since the reverse π^{R} of a path π is a path, paths always come in pairs. We really think of π and π^{R} as the same object, but we have to distinguish them formally in order to talk about how different paths correspond to each other.[5]

A *maximal path* is a path that is not a proper subpath of any other path. A maximal path starts and ends either inside the endsequent or inside the principal formula of an application of Weakening. In a cut-free derivation, at least one of the endpoints of a maximal path must be inside the endsequent. A *cycle* is a path that starts and ends in the same occurrence of a propositional variable. It is easy to see that no cycle can occur in a cut-free derivation (cf. Carbone 1997).

Consider the following reduction steps for cut elimination:[6]

[5] Carbone's (1997) notion of *logical path* is designed to pick out one path from each pair $\{\pi, \pi^{\mathrm{R}}\}$.

[6] These reduction steps are found in Borisavljević 1999, modulo the absence of Interchange.

(C1)
$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow C} \quad \overset{\mathscr{D}_2}{D, \Delta \Rightarrow A}}{C \to D, \Gamma, \Delta \Rightarrow A}(\to\Rightarrow) \quad \overset{\mathscr{D}_3}{A, \Theta \Rightarrow B}}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow C} \quad \dfrac{\overset{\mathscr{D}_2}{D, \Delta \Rightarrow A} \quad \overset{\mathscr{D}_3}{A, \Theta \Rightarrow B}}{D, \Delta, \Theta \Rightarrow B}\text{Cut}}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}(\to\Rightarrow)$$

(C2)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{A, \Delta \Rightarrow C} \quad \overset{\mathscr{D}_3}{D, \Theta \Rightarrow B}}{C \to D, A, \Delta, \Theta \Rightarrow B}(\to\Rightarrow)}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow C}}{\Gamma, \Delta \Rightarrow C}\text{Cut} \quad \overset{\mathscr{D}_3}{D, \Theta \Rightarrow B}}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}(\to\Rightarrow)$$

(C3)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow C} \quad \overset{\mathscr{D}_3}{D, A, \Theta \Rightarrow B}}{C \to D, A, \Delta, \Theta \Rightarrow B}(\to\Rightarrow)}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow C} \quad \dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_3}{D, A, \Theta \Rightarrow B}}{D, \Gamma, \Theta \Rightarrow C}\text{Cut}}{C \to D, \Gamma, \Delta, \Theta \Rightarrow B}(\to\Rightarrow)$$

(C4)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{C, A, \Delta \Rightarrow D}}{A, \Delta \Rightarrow C \to D}(\Rightarrow\to)}{\Gamma, \Delta \Rightarrow C \to D}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{C, A, \Delta \Rightarrow D}}{C, \Gamma, \Delta \Rightarrow D}\text{Cut}}{\Gamma, \Delta \Rightarrow C \to D}(\Rightarrow\to)$$

(C5)
$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{A, \Gamma \Rightarrow B}}{\Gamma \Rightarrow A \to B}(\Rightarrow\to) \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow A} \quad \overset{\mathscr{D}_3}{B, \Theta \Rightarrow C}}{A \to B, \Delta, \Theta \Rightarrow C}(\to\Rightarrow)}{\Gamma, \Delta, \Theta \Rightarrow C}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_1}{A, \Gamma \Rightarrow B} \quad \overset{\mathscr{D}_3}{B, \Theta \Rightarrow C}}{A, \Gamma, \Theta \Rightarrow C}\text{Cut}}{\Delta, \Gamma, \Theta \Rightarrow C}\text{Cut}$$

(C6)
$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{C, C, \Gamma \Rightarrow A}}{C, \Gamma \Rightarrow A}\text{Contr} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{C, \Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{C, C, \Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{C, C, \Gamma, \Delta \Rightarrow B}\text{Cut}}{C, \Gamma, \Delta \Rightarrow B}\text{Contr}$$

(C7)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{C, C, A, \Delta \Rightarrow B}}{C, A, \Delta \Rightarrow B}\text{Contr}}{C, \Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{C, C, A, \Delta \Rightarrow B}}{C, C, \Gamma, \Delta \Rightarrow B}\text{Cut}}{C, \Gamma, \Delta \Rightarrow B}\text{Contr}$$

(C8)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{A, A, \Delta \Rightarrow B}}{A, \Delta \Rightarrow B}\text{Contr}}{\Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{A, A, \Delta \Rightarrow B}}{A, \Gamma, \Delta \Rightarrow B}\text{Cut}}{\Gamma, \Delta \Rightarrow B}\text{Cut}\dagger$$

(C9)
$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A}}{C, \Gamma \Rightarrow A}\text{Weak} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{C, \Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{\Gamma, \Delta \Rightarrow B}\text{Cut}}{C, \Gamma, \Delta \Rightarrow B}\text{Weak}$$

(C10)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{C, A, \Delta \Rightarrow B}\text{Weak}}{C, \Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \overset{\mathscr{D}_2}{A, \Delta \Rightarrow B}}{\Gamma, \Delta \Rightarrow B}\text{Cut}}{C, \Gamma, \Delta \Rightarrow B}\text{Weak}$$

(C11)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow B}}{A, \Delta \Rightarrow B}\text{Weak}}{\Gamma, \Delta \Rightarrow B}\text{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow B}}{\Gamma, \Delta \Rightarrow B}\text{Weak}$$

(C12)
$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad A \Rightarrow A}{\Gamma \Rightarrow A}\text{Cut} \quad \rightsquigarrow \quad \overset{\mathscr{D}_1}{\Gamma \Rightarrow A}$$

(C13)
$$\dfrac{A \Rightarrow A \quad \overset{\mathscr{D}_1}{A, \Gamma \Rightarrow A}}{A, \Gamma \Rightarrow A}\text{Cut} \quad \rightsquigarrow \quad \overset{\mathscr{D}_1}{A, \Gamma \Rightarrow A}$$

Except for (C8) and (C11), there is a one-one correspondence between the maximal paths in the original derivation and the derivation after the reduction. Let us write $\mathscr{D} \twoheadrightarrow_\flat \mathscr{D}'$ just in case $\mathscr{D}$ reduces to $\mathscr{D}'$ by repeated applications of (C1), (C2), (C3), (C4), (C5), (C6), (C7), (C9), (C10), (C12), (C13). Since this notion of reduction does not involve the problematic case (C8) of cut-elimination, it is clear that the converse of the relation $\twoheadrightarrow_{\flat} \cap \neq$ is well-founded, i.e., every reduction sequence terminates.

Definition 3. A cut-free derivation in $LJ_{\to}$ is *W-normal* if none of the following reduction steps is applicable to it:[7]

$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A}}{D,\Gamma \Rightarrow A}\,\text{Weak} \qquad \overset{\mathscr{D}_2}{B,\Delta \Rightarrow C}}{A \to B, D, \Gamma, \Delta \Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma\Rightarrow A}\quad \overset{\mathscr{D}_2}{B,\Delta\Rightarrow C}}{A\to B,\Gamma,\Delta\Rightarrow C}\,(\to\Rightarrow)}{D, A\to B,\Gamma,\Delta\Rightarrow C}\,\text{Weak}$$

$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{B,\Delta \Rightarrow C}}{D,B,\Delta \Rightarrow C}\,\text{Weak}}{A \to B, \Gamma, D, \Delta \Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma\Rightarrow A}\quad \overset{\mathscr{D}_2}{B,\Delta\Rightarrow C}}{A\to B,\Gamma,\Delta\Rightarrow C}\,(\to\Rightarrow)}{D, A\to B,\Gamma,\Delta\Rightarrow C}\,\text{Weak}$$

$$\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow A} \quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow C}}{B,\Delta \Rightarrow C}\,\text{Weak}}{A \to B, \Gamma, \Delta \Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\overset{\mathscr{D}_2}{\Delta \Rightarrow C}}{A\to B,\Gamma,\Delta\Rightarrow C}\,\text{Weak}$$

$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{A,\Gamma \Rightarrow B}}{C,A,\Gamma \Rightarrow B}\,\text{Weak}}{C,\Gamma \Rightarrow A \to B}\,(\Rightarrow\to) \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{A,\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \to B}\,(\Rightarrow\to)}{C,\Gamma \Rightarrow A \to B}\,\text{Weak}$$

$$\dfrac{\dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow B}}{C,\Gamma \Rightarrow B}\,\text{Weak}}{A,C,\Gamma \Rightarrow B}\,\text{Weak}}{C,\Gamma \Rightarrow A \to B}\,(\Rightarrow\to) \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\mathscr{D}_1}{\Gamma \Rightarrow B}}{A,\Gamma \Rightarrow B}\,\text{Weak}}{\Gamma \Rightarrow A \to B}\,(\Rightarrow\to)}{C,\Gamma \Rightarrow A \to B}\,\text{Weak}$$

$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{A,A,\Gamma \Rightarrow B}}{C,A,A,\Gamma \Rightarrow B}\,\text{Weak}}{C,A,\Gamma \Rightarrow B}\,\text{Contr} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{A,A,\Gamma \Rightarrow B}}{A,\Gamma \Rightarrow B}\,\text{Contr}}{C,A,\Gamma \Rightarrow B}\,\text{Weak}$$

$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{A,\Gamma \Rightarrow B}}{A,A,\Gamma \Rightarrow B}\,\text{Weak}}{A,\Gamma \Rightarrow B}\,\text{Contr} \quad\rightsquigarrow\quad \overset{\mathscr{D}_1}{A,\Gamma \Rightarrow B}$$

Every cut-free derivation can be put into a W-normal form with the same λ-term by repeatedly applying these reduction steps.

Definition 4. A cut-free derivation in $LJ_{\to}$ is *WC-normal* if it is W-normal and moreover if none of the following reduction steps is applicable to it:[8]

$$\dfrac{\dfrac{\overset{\mathscr{D}_1}{D,D,\Gamma \Rightarrow A}}{D,\Gamma \Rightarrow A}\,\text{Contr} \qquad \overset{\mathscr{D}_2}{B,\Delta \Rightarrow C}}{A \to B, D, \Gamma, \Delta \Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\mathscr{D}_1}{D,D,\Gamma\Rightarrow A}\quad \overset{\mathscr{D}_2}{B,\Delta\Rightarrow C}}{A\to B,D,D,\Gamma,\Delta\Rightarrow C}\,(\to\Rightarrow)}{A\to B,D,\Gamma,\Delta\Rightarrow C}\,\text{Contr}$$

[7]The present definition is not exactly the same as that found in Mints 1996 (restricted to the implicational fragment), since the latter uses the additive version of $(\Rightarrow\to)$.

[8]Again this definition is slightly different from Mints 1996 due to the difference in the formulation of $(\to\Rightarrow)$.

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ \Gamma\Rightarrow A\end{matrix}\quad \dfrac{\begin{matrix}\mathscr{D}_2\\ B,D,D,\Delta\Rightarrow C\end{matrix}}{B,D,\Delta\Rightarrow C}\,\text{Contr}}{A\to B,\Gamma,D,\Delta\Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ \Gamma\Rightarrow A\end{matrix}\quad \begin{matrix}\mathscr{D}_2\\ B,D,D,\Delta\Rightarrow C\end{matrix}}{A\to B,\Gamma,D,D,\Delta\Rightarrow C}\,(\to\Rightarrow)}{A\to B,\Gamma,D,\Delta\Rightarrow C}\,\text{Contr}$$

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ \Gamma\Rightarrow A\end{matrix}\quad \dfrac{\begin{matrix}\mathscr{D}_2\\ B,B,\Delta\Rightarrow C\end{matrix}}{B,\Delta\Rightarrow C}\,\text{Contr}}{A\to B,\Gamma,\Delta\Rightarrow C}\,(\to\Rightarrow) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ \Gamma\Rightarrow A\end{matrix}\quad \dfrac{\begin{matrix}\mathscr{D}_1\\ \Gamma\Rightarrow A\end{matrix}\quad \begin{matrix}\mathscr{D}_2\\ B,B,\Delta\Rightarrow C\end{matrix}}{A\to B,\Gamma,B,\Delta\Rightarrow C}\,(\to\Rightarrow)}{A\to B,A\to B,\Gamma,\Delta\Rightarrow C}\,(\to\Rightarrow)\dagger}{A\to B,\Gamma,\Delta\Rightarrow C}\,\text{Contr}$$

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ C,C,A,\Gamma\Rightarrow B\end{matrix}}{C,A,\Gamma\Rightarrow B}\,\text{Contr}}{C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ C,C,A,\Gamma\Rightarrow B\end{matrix}}{C,C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to)}{C,\Gamma\Rightarrow A\to B}\,\text{Contr}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ C,C,\Gamma\Rightarrow B\end{matrix}}{C,\Gamma\Rightarrow B}\,\text{Contr}}{A,C,\Gamma\Rightarrow B}\,\text{Weak}}{C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to) \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ C,C,\Gamma\Rightarrow B\end{matrix}}{A,C,C,\Gamma\Rightarrow B}\,\text{Weak}}{C,C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to)}{C,\Gamma\Rightarrow A\to B}\,\text{Contr}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A^n,C,C,\Gamma\Rightarrow B\end{matrix}}{A^n,C,\Gamma\Rightarrow B}\,\text{Contr}}{A,C,\Gamma\Rightarrow B}\,\text{Contr}}{C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to) \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A^n,C,C,\Gamma\Rightarrow B\end{matrix}}{A,C,C,\Gamma\Rightarrow B}\,\text{Contr}}{C,C,\Gamma\Rightarrow A\to B}\,(\Rightarrow\to)}{C,\Gamma\Rightarrow A\to B}\,\text{Contr}$$

Every cut-free W-normal derivation can be put into a WC-normal form with the same λ-term by repeatedly applying these reduction steps.

2.2 Maehara's method

Maehara's (1960) method is the most commonly used syntactical method for proving interpolation (see Troelstra and Schwichtenberg 2000). We reformulate it using Wroński's (1984) idea of using sequences of formulas in place of single interpolation formulas, and prove that the method satisfies stronger conditions than those stated by the usual form of the Interpolation Theorem.

Notations. We will often have to refer to a large number of sequences, which necessitates compact notations for representing them. In what follows, we will use the following abbreviatory conventions:

$\mathbf{e}_1^n$	abbreviates	$\mathbf{e}_1,\dots,\mathbf{e}_n$,	where $\mathbf{e}$ is a letter (possibly with diacritics).
$(\mathbf{e}[i])_{i=1}^n$	abbreviates	$\mathbf{e}[1],\dots,\mathbf{e}[n]$,	where $\mathbf{e}[i]$ is an expression containing i.
$(\mathbf{e}[i])_{i\in S}$	abbreviates	$\mathbf{e}[s_1],\dots,\mathbf{e}[s_n]$,	where $\mathbf{e}[i]$ is as above and $s_1,\dots,s_n$ lists the elements of S in increasing order.
$\vec{A}\to B$	abbreviates	$A_1\to\cdots\to A_n\to B$	if $\vec{A}$ represents $A_1,\dots,A_n$.

If R is a two-premise rule and $\vec{\mathscr{D}}$ represents $\mathscr{D}_1,\dots,\mathscr{D}_n$,

$$\dfrac{\mathscr{D}_0\quad \vec{\mathscr{D}}}{\Gamma\Rightarrow C}\,R \quad\text{abbreviates}\quad \dfrac{\begin{matrix}\dfrac{\mathscr{D}_0\quad\mathscr{D}_1}{}\,R\\ \vdots\end{matrix}\qquad \mathscr{D}_n}{\Gamma\Rightarrow C}\,R$$

$$\dfrac{\vec{\mathscr{D}} \quad \mathscr{D}_0}{\Gamma \Rightarrow C}\, R \quad \text{abbreviates} \quad \dfrac{\mathscr{D}_1 \qquad \dfrac{\mathscr{D}_n \quad \mathscr{D}_0}{\vdots}\, R}{\Gamma \Rightarrow C}\, R$$

If $\mathbf{e}$ is any expression, we use

$$(\mathbf{e})^\circ$$

as a metavariable whose value is either an empty expression or $\mathbf{e}$. When we use the same expression $(\mathbf{e})^\circ$ more than once, the different occurrences of $(\mathbf{e})^\circ$ are not necessarily intended to stand for the same thing.

Definition 5. Let $\Gamma, \Delta \Rightarrow T : C$ be a sequent such that T is normal. A sequence of cut-free W-normal derivations $(\mathscr{D}_i \colon \Gamma_i \Rightarrow S_i : E_i)_{i=1}^m$ is said to be an *$LJ_\to$-interpolant to* $\Gamma, \Delta \Rightarrow T : C$ *with respect to the partition* $(\Gamma; \Delta)$, if there exists a cut-free W-normal derivation $\mathscr{D}_0 \colon (z_i : E_i)_{i=1}^m, \Delta_0 \Rightarrow P : C$ such that the following conditions hold:[9]

1. $\Gamma_i \subseteq \Gamma$ $(i = 1, \ldots, m)$;
2. $\Delta_0 \subseteq \Delta$;
3. $P[(S_i/z_i)_{i=1}^m] \twoheadrightarrow_\beta T$;
4. In $\mathscr{D}_i$ $(i = 1, \ldots, m)$, every maximal path starting inside the succedent E_i of the endsequent ends inside the antecedent Γ_i of the endsequent;
5. In $\mathscr{D}_0$, every maximal path starting inside $(z_i : E_i)_{i=1}^m$ in the endsequent ends inside Δ_0 or C in the endsequent.

In this case, we call $\mathscr{D}_0$ an *auxiliary derivation* for $\mathscr{D}_1^m, \mathscr{D}$, and we say that $\mathscr{D}_1^m$ is an interpolant to $\Gamma, \Delta \Rightarrow T : C$ (with respect to the partition $(\Gamma; \Delta)$) *via* $\mathscr{D}_0$.

Theorem 6. *Given a cut-free derivation* $\mathscr{D} \colon \Gamma, \Delta \Rightarrow T : C$*, one can find an $LJ_\to$-interpolant to* $\Gamma, \Delta \Rightarrow T : C$ *with respect to the partition* $(\Gamma; \Delta)$.

If Γ is a context, we let Γ^- denote the multiset of formulas which is the result of deleting all variables and colons from Γ. It is easy to see that Theorem 6 implies the modification of the usual statement of the Interpolation Theorem mentioned in section 1:

Lemma 7. *If* $(\mathscr{D}_i \colon \Gamma_i \Rightarrow S_i : E_i)_{i=1}^m$ *is an $LJ_\to$-interpolant to* $\Gamma, \Delta \Rightarrow T : C$ *with respect to the partition* $(\Gamma; \Delta)$*, then* $E_1, \ldots, E_m$ *is an interpolation sequence for* $\Gamma^-, \Delta^- \Rightarrow C$ *with respect to the partition* $(\Gamma^-; \Delta^-)$.

Proof of Theorem 6. We construct cut-free W-normal derivations $(\mathscr{D}_i \colon \Gamma_i \Rightarrow S_i : E_i)_{i=1}^m, \mathscr{D}_0 \colon (z_i : E_i)_{i=1}^m, \Delta_0 \Rightarrow P : C$ satisfying the conditions 1–5 of Definition 5 by induction on cut-free derivation $\mathscr{D} \colon \Gamma, \Delta \Rightarrow T : C$. We choose to construct at each step W-normal derivations that do not end in Weakening.

Induction Basis. $\mathscr{D}$ is an initial sequent $x : A \Rightarrow x : A$. *Case 1.* $\Gamma = x : A$ and $\Delta = \varnothing$. Then we can take $m = 1$, $\mathscr{D}_1 = \mathscr{D}$, and $\mathscr{D}_0 = z_1 : A \Rightarrow z_1 : A$. *Case 2.* $\Gamma = \varnothing$ and $\Delta = x : A$. Then we can take $m = 0$, $\mathscr{D}_0 = \mathscr{D}$. In both cases, conditions 1–5 are clearly satisfied.

[9] We require W-normality so that $LJ_\to$-interpolants translate into interpolants in natural deduction.

Induction Step.

Case 1. The last inference of $\mathscr{D}$ is $(\to\Rightarrow)$:

$$\dfrac{\overset{\mathscr{D}'}{\Gamma', \Delta' \Rightarrow U : A} \quad \overset{\mathscr{D}''}{y : B, \Gamma'', \Delta'' \Rightarrow Q : C}}{x : A \to B, \Gamma', \Gamma'', \Delta', \Delta'' \Rightarrow Q[xU/y] : C}\ (\to\Rightarrow)$$

where $\Gamma', \Gamma'' \subseteq \Gamma$ and $\Delta', \Delta'' \subseteq \Delta$. There are two subcases depending on whether $x : A \to B$ is in Γ.

Case 1.1. $\Gamma = x : A \to B, \Gamma', \Gamma''$ and $\Delta = \Delta', \Delta''$. We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Delta'; \Gamma')$ and to $\mathscr{D}''$ with respect to the partition $(y : B, \Gamma''; \Delta'')$. From $\mathscr{D}'$, we obtain $n \geq 0$, $(\mathscr{D}'_i : \Delta'_i \Rightarrow S'_i : F_i)_{i=1}^n$, $\mathscr{D}_0 : (w_i : F_i)_{i=1}^n, \Gamma'_0 \Rightarrow P' : A$ satisfying the required properties, namely:

(1) i. $\Delta'_i \subseteq \Delta'$ $(i = 1, \dots, n)$;
ii. $\Gamma'_0 \subseteq \Gamma'$;
iii. $P'[(S'_i/w_i)_{i=1}^n] \twoheadrightarrow_\beta U$;
iv. In $\mathscr{D}'_i$ $(i = 1, \dots, n)$, every maximal path starting inside the succedent F_i of the endsequent ends inside the antecedent Δ'_i of the endsequent.
v. In $\mathscr{D}'_0$, every maximal path starting inside $(w_i : F_i)_{i=1}^n$ in the endsequent ends inside Γ'_0 or A in the endsequent.

From $\mathscr{D}''$, we obtain $p \geq 0$, $(\mathscr{D}''_i : \Theta_i, \Gamma''_i \Rightarrow S''_i{:}G_i)_{i=1}^p$, $\mathscr{D}''_0 : (v_i{:}G_i)_{i=1}^p, \Delta''_0 \Rightarrow P''{:}C$ satisfying the required properties, namely:

(2) i. $\Theta_i \subseteq y : B$ and $\Gamma''_i \subseteq \Gamma''$ $(i = 1, \dots, p)$;
ii. $\Delta''_0 \subseteq \Delta''$;
iii. $P''[(S''_i/v_i)_{i=1}^p] \twoheadrightarrow_\beta Q$;
iv. In $\mathscr{D}''_i$ $(i = 1, \dots, p)$, every maximal path starting inside the succedent G_i of the endsequent ends inside the antecedent Θ_i, Γ''_i of the endsequent.
v. In $\mathscr{D}''_0$, every maximal path starting inside $(v_i : G_i)_{i=1}^p$ in the endsequent ends inside Δ''_0 or C in the endsequent.

Let

$$\begin{aligned} P^+ &= \{\, i \mid 1 \leq i \leq p, \Theta_i = y : B \,\}, \\ P^- &= \{1, \dots, p\} - P^+. \end{aligned}$$

Let $m = p$, and let

$$\mathscr{D}_i = \begin{cases} \dfrac{\dfrac{\overset{\mathscr{D}'_0}{(w_i : F_i)_{i=1}^n, \Gamma'_0 \Rightarrow P' : A} \quad \overset{\mathscr{D}''_i}{y : B, \Gamma''_i \Rightarrow S''_i : G_i}}{x : A \to B, (w_i : F_i)_{i=1}^n, \Gamma'_0, \Gamma''_i \Rightarrow S''_i[xP'/y] : G_i}\ (\to\Rightarrow)}{x : A \to B, \Gamma'_0, \Gamma''_i \Rightarrow \lambda w_1^n . S''_i[xP'/y] : F_1^n \to G_i}\ (\Rightarrow\to) & \text{for } i \in P^+, \\[2ex] \mathscr{D}''_i & \text{for } i \in P^-, \end{cases}$$

$$\mathscr{D}_0 = \dfrac{\left(\left(\overset{\mathscr{D}'_i}{\Delta'_i \Rightarrow S'_i : F_i}\right)_{i=1}^n\right)^{|P^+|} \quad \overset{\mathscr{D}''_0}{(v_i : G_i)_{i=1}^p, \Delta''_0 \Rightarrow P'' : C}}{(z_i : F_1^n \to G_i)_{i \in P^+}, (v_i : G_i)_{i \in P^-}, \Delta'_1 \cup \dots \cup \Delta'_n, \Delta''_0 \Rightarrow P''[(z_i S'^{\,n}_1/v_i)_{i \in P^+}] : C}\ (\to\Rightarrow)\dagger$$

We show that conditions 1–5 hold of $\mathscr{D}_1^p$ and $\mathscr{D}_0$. For condition 1, we have

$$\begin{aligned}(x : A \to B, \Gamma_0')^\circ, \Gamma_i'' &\subseteq x : A \to B, \Gamma', \Gamma'' \quad \text{by (1.ii) and (2.i)} \\ &= \Gamma\end{aligned}$$

For condition 2, we have

$$\begin{aligned}\Delta_1' \cup \cdots \cup \Delta_n', \Delta_0'' &\subseteq \Delta', \Delta'' \quad \text{by (1.i) and (2.ii)} \\ &= \Delta\end{aligned}$$

For condition 3, we have

$$\begin{aligned}&P''[(z_i S'^n_1/v_i)_{i\in P^+}][(\lambda w_1^n.S_i''[xP'/y]/z_i)_{i\in P^+}, (S_i''/v_i)_{i\in P^-}] \\ &\qquad = P''[((\lambda w_1^n.S_i''[xP'/y])S'^n_1/v_i)_{i\in P^+}][(S_i''/v_i)_{i\in P^-}] \\ &\qquad \twoheadrightarrow_\beta P''[(S_i''[xP'/y][(S_i'/w_i)_{i=1}^n]/v_i)_{i\in P^+}][(S_i''/v_i)_{i\in P^-}] \\ &\qquad = P''[(S_i''[xP'[(S_i'/w_i)_{i=1}^n]/y]/v_i)_{i\in P^+}][(S_i''/v_i)_{i\in P^-}] \\ &\qquad \twoheadrightarrow_\beta P''[(S_i''[xU/y]/v_i)_{i\in P^+}][(S_i''/v_i)_{i\in P^-}] \quad \text{by (1.iii)} \\ &\qquad = P''[(S_i''/v_i)_{i\in P^+}][(S_i''/v_i)_{i\in P^-}][xU/y] \\ &\qquad \twoheadrightarrow_\beta Q[xU/y] \quad \text{by (2.iii).}\end{aligned}$$

Condition 4 holds of $\mathscr{D}_i$ for $i \in P^-$ by (2.iv). To see that condition 4 holds of $\mathscr{D}_i$ for $i \in P^+$, note that any maximal path in $\mathscr{D}_i$ starting inside $F_1^n \to G_i$ in the endsequent must pass through an occurrence inside $w_j : F_j$ in the endsequent of $\mathscr{D}_0'$ or an occurrence inside G_i in the endsequent of $\mathscr{D}_i''$. In the former case, (1v) ensures that it must reach an occurrence inside Γ_0' or A in the endsequent of $\mathscr{D}_0'$, from where it reaches an occurrence inside $x : A \to B$ or Γ_0' in the endsequent of $\mathscr{D}_i$, terminating there. In the latter case, (2.iv) ensures that the path goes through an occurrence inside $y : B, \Gamma_i''$ in the endsequent of $\mathscr{D}_i''$, and it eventually ends up inside $x : A \to B$ or Γ_i'' in the endsequent of $\mathscr{D}_i$.

Finally, let us show that condition 5 holds of $\mathscr{D}_0$. Any maximal path in $\mathscr{D}_0$ starting inside $(v_i : G_i)_{i\in P^-}$ in the endsequent of $\mathscr{D}_0$ must reach an occurrence inside $(v_i : G_i)_{i\in P^-}$ in the endsequent of $\mathscr{D}_0''$, from which it reaches an occurrence inside Δ_0'' or C in the endsequent of $\mathscr{D}_0''$ by (2.iv). The path then ends inside Δ_0'' or C in the endsequent of $\mathscr{D}_0$. If a maximal path in $\mathscr{D}_0$ starts in an occurrence inside $(z_i : F_1^n \to G_i)_{i\in P^+}$, it must reach an occurrence inside F_i in the endsequent of $\mathscr{D}_i'$ or an occurrence inside $(v_i : G_i)_{i=1}^p$ in the endsequent of $\mathscr{D}_0''$. In the former case, it reaches an occurrence inside Δ_i' in the endsequent of $\mathscr{D}_i'$ by (1.iv), and ends up inside Δ_i' in the endsequent of $\mathscr{D}_0$. In the latter case, the path reaches an occurrence inside Δ_0'' or C in the endsequent of $\mathscr{D}_0''$ by (2v), and ends up inside an occurrence inside Δ_0'' or C in the endsequent of $\mathscr{D}_0$.

Case 1.2. $\Gamma = \Gamma', \Gamma''$ and $\Delta = x : A \to B, \Delta', \Delta''$. We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Gamma'; \Delta')$ and to $\mathscr{D}''$ with respect to the partition $(\Gamma''; y : B, \Delta'')$. From $\mathscr{D}'$, we obtain $n \geq 0$, $(\mathscr{D}_i' : \Gamma_i' \Rightarrow S_i' : F_i)_{i=1}^n$, $\mathscr{D}_0' : (w_i : F_i)_{i=1}^n, \Delta_0' \Rightarrow P' : A$ satisfying the required properties. From $\mathscr{D}''$, we obtain $p \geq 0$, $(\mathscr{D}_i'' : \Gamma_i'' \Rightarrow S_i'' : G_i)_{i=1}^p$, and $\mathscr{D}_0'' : (v_i : G_i)_{i=1}^p, \Theta, \Delta_0'' \Rightarrow P'' : C$ satisfying the required properties, where $\Theta \subseteq y : B$ and $\Delta_0'' \subseteq \Delta''$.

We distinguish two subcases according to whether $y : B \in \Theta$.

Case 1.2.1. $\Theta = y : B$. Let $m = n + p$, and let

$$\mathscr{D}_i = \begin{cases} \mathscr{D}'_i & \text{for } i = 1, \dots, n, \\ \mathscr{D}''_{i-n} & \text{for } i = n+1, \dots, n+p, \end{cases}$$

$$\mathscr{D}_0 = \cfrac{\overset{\mathscr{D}'_0}{(w_i : F_i)_{i=1}^n, \Delta'_0 \Rightarrow P' : A} \quad \overset{\mathscr{D}''_0}{(v_i : G_i)_{i=1}^r, y : B, \Delta''_0 \Rightarrow P'' : C}}{(w_i : F_i)_{i=1}^n, (z_i : G_i)_{i=1}^p, x : A \to B, \Delta'_0, \Delta''_0 \Rightarrow P''[xP'/y] : C} \; (\to\Rightarrow)$$

It is easy to see that conditions 1–5 hold of $\mathscr{D}_1^{n+p}, \mathscr{D}_0$. We leave the proof of correctness to the reader here as well as in the remaining cases.

Case 1.2.2. $\Theta = \varnothing$. Let $m = p$, and let

$$\mathscr{D}_i = \mathscr{D}''_i \quad \text{for } i = 1, \dots, p$$
$$\mathscr{D}_0 = \mathscr{D}''_0.$$

Case 2. The last inference of $\mathscr{D}$ is $(\Rightarrow\to)$:

$$\cfrac{\overset{\mathscr{D}'}{x : A, \Gamma, \Delta \Rightarrow Q : B}}{\Gamma, \Delta \Rightarrow \lambda x.Q : A \to B} \; (\Rightarrow\to)$$

We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Gamma; x : A, \Delta)$. We obtain $n \geq 0$, $(\mathscr{D}'_i : \Gamma_i \Rightarrow S'_i : F_i)_{i=1}^n, \mathscr{D}'_0 : (w_i : F_i)_{i=1}^n, \Theta, \Delta_0 \Rightarrow P' : B$ satisfying the required properties, where $\Theta \subseteq x : A$ and $\Delta_0 \subseteq \Delta$.

Let $m = n$, and let $\mathscr{D}_i = \mathscr{D}'_i$ for $i = 1, \dots, n$. As for $\mathscr{D}_0$, we distinguish two subcases.

Case 2.1. $\Theta = x : A$. Let

$$\mathscr{D}_0 = \cfrac{\overset{\mathscr{D}'_0}{(w_i : F_i)_{i=1}^n, x : A, \Delta_0 \Rightarrow P' : B}}{(w_i : F_i)_{i=1}^n, \Delta_0 \Rightarrow \lambda x.P' : A \to B} \; (\to\Rightarrow)$$

Case 2.2. $\Theta = \varnothing$. Let

$$\mathscr{D}_0 = \cfrac{\cfrac{\overset{\mathscr{D}'_0}{(w_i : F_i)_{i=1}^n, \Delta_0 \Rightarrow P' : B}}{(w_i : F_i)_{i=1}^n, x : A, \Delta_0 \Rightarrow P' : B} \; \text{Weak}}{(w_i : F_i)_{i=1}^n, \Delta_0 \Rightarrow \lambda x.P' : A \to B} \; (\to\Rightarrow)$$

Case 3. The last inference of $\mathscr{D}$ is Contraction:

$$\cfrac{\overset{\mathscr{D}'}{y : A, z : A, \Gamma', \Delta' \Rightarrow Q : C}}{x : A, \Gamma', \Delta' \Rightarrow Q[x/y, x/z] : C} \; \text{Contr}$$

where $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. There are two subcases depending on whether $x : A \in \Gamma$.

Case 3.1. $\Gamma = x : A, \Gamma'$ and $\Delta = \Delta'$. We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(y : A, z : A, \Gamma'; \Delta')$. We obtain $n \geq 0$, $(\mathscr{D}'_i : \Theta_i, \Gamma'_i \Rightarrow S'_i : F_i)_{i=1}^n$, $\mathscr{D}'_0 : (w_i : F_i)_{i=1}^n, \Delta'_0 \Rightarrow P' : C$ satisfying the required properties, where

$$\Theta_i \subseteq y : A, z : A$$
$$\Gamma'_i \subseteq \Gamma'$$

for $i = 1, \dots, n$.

Let $m = p$, let

$$\mathscr{D}_i = \begin{cases} \dfrac{\begin{array}{c}\mathscr{D}_i' \\ y : A, z : A, \Gamma_i' \Rightarrow S_i' : F\end{array}}{x : A, \Gamma_i' \Rightarrow S_i'[x/y, x/z] : F_i}\ \text{Contr} & \text{if } \Theta_i = y : A, z : A, \\[2ex] \begin{array}{c}\mathscr{D}_i'[x/y, x/z] \\ \Gamma_i'[x/y, x/z] \Rightarrow S_i'[x/y, x/z] : F_i\end{array} & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, n$, and let

$$\mathscr{D}_0 = \mathscr{D}_0'.$$

Case 3.2. $\Gamma = \Gamma'$ and $\Delta = x : A, \Delta'$. We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Gamma'; y : A, z : A, \Delta')$. We obtain $n \geq 0$, $(\mathscr{D}_i' : \Gamma_i' \Rightarrow S_i' : F_i)_{i=1}^n$, $\mathscr{D}_0' : (w_i : F_i)_{i=1}^n, \Theta, \Delta_0' \Rightarrow P' : C$ satisfying the required properties, where $\Theta \subseteq y : A, z : A$ and $\Delta_0' \subseteq \Delta'$.

Let $m = n$ and let

$$\mathscr{D}_i = \mathscr{D}_i' \quad \text{for } i = 1, \dots, n.$$

As for $\mathscr{D}_0$, we distinguish two subcases.

Case 3.2.1. $\Theta = y : A, z : A$. Let

$$\mathscr{D}_0 = \dfrac{\begin{array}{c}\mathscr{D}_0' \\ y : A, z : A, \Delta_0' \Rightarrow P' : C\end{array}}{x : A, \Delta_0' \Rightarrow P'[x/y, x/z] : C}\ \text{Contr}$$

Case 3.2.2. $\Theta \subsetneq y : A, z : A$. Let

$$\mathscr{D}_0 = \begin{array}{c}\mathscr{D}_0'[x/y, x/z] \\ \Theta[x/y, x/z], \Delta_0' \Rightarrow P'[x/y, x/z] : C\end{array}$$

Case 4. The last inference of $\mathscr{D}$ is Weakening:

$$\dfrac{\begin{array}{c}\mathscr{D}' \\ \Gamma', \Delta' \Rightarrow T : C\end{array}}{x : A, \Gamma', \Delta' \Rightarrow T : C}\ \text{Weak}$$

where $\Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta$. We apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Gamma'; \Delta')$ and obtain $n \geq 0$, $(\mathscr{D}_i' : \Gamma_i' \Rightarrow S_i' : F_i)_{i=1}^n$, and $\mathscr{D}_0' : (w_i : F_i)_{i=1}^n, \Delta_0' \Rightarrow P' : C$ satisfying the required properties.

Let $m = n$, and let

$$\begin{aligned} \mathscr{D}_i &= \mathscr{D}_i' \quad \text{for } i = 1, \dots, n, \\ \mathscr{D}_0 &= \mathscr{D}_0'. \end{aligned}$$

□

Remark. The input derivation $\mathscr{D}$ to the above method can be first turned into a W-normal derivation without affecting the output derivations $\mathscr{D}_1^m, \mathscr{D}_0$.

We can ascribe to the output derivations of Maehara's method a slightly stronger condition than condition 3 of Definition 5:

Theorem 8. *Suppose that, given* $\mathscr{D}\colon \Gamma, \Delta \Rightarrow T : C$ *and partition* $(\Gamma; \Delta)$*, Maehara's method returns* $(\mathscr{D}_i\colon \Gamma_i \Rightarrow S_i : E_i)_{i=1}^m, \mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta_0 \Rightarrow P : C$*. Let*

$$\mathscr{C} = \cfrac{\cfrac{\left(\begin{array}{c}\mathscr{D}_i \\ \Gamma_i \Rightarrow S_i : E_i\end{array}\right)_{i=1}^m \quad \begin{array}{c}\mathscr{D}_0 \\ (z_i : E_i)_{i=1}^m, \Delta_0 \Rightarrow P : C\end{array}}{\Gamma_1 \cup \cdots \cup \Gamma_m, \Delta_0 \Rightarrow P[(S_i/z_i)_{i=1}^m] : C}\ \mathit{Cut}\dagger}{\Gamma, \Delta \Rightarrow P[(S_i/z_i)_{i=1}^m] : C}\ \mathit{Weak}$$

Then $\mathscr{C} \twoheadrightarrow_\flat \hat{\mathscr{D}}$ *for some cut-free W-normal derivation* $\hat{\mathscr{D}}\colon \Gamma, \Delta \Rightarrow T : C$*.*

Proof. It suffices to show that in $\mathscr{D}_1^m$ and $\mathscr{D}_0$, no subformula of E_i in the endsequent has an ancestor which is a principal formula of Contraction or Weakening. This can be checked by induction easily. ◻

The above theorem does not necessarily hold with $\hat{\mathscr{D}} = \mathscr{D}$, even when $\mathscr{D}$ is *W*-normal. However, we can show the following:

Theorem 9. *Let* $\mathscr{D}, \mathscr{D}_1^m, \mathscr{D}_0, \mathscr{C}$ *be as in Theorem 8. If* $\mathscr{D}$ *is WC-normal, then* $\mathscr{D}_1^m, \mathscr{D}_0$ *are all WC-normal, and* $\mathscr{C} \twoheadrightarrow_\flat \hat{\mathscr{D}}$ *for some WC-normal derivation* $\hat{\mathscr{D}}\colon \Gamma, \Delta \Rightarrow T : C$ *that is identical to* $\mathscr{D}$ *modulo reordering within the final block of applications of Contraction.*

Proof. The theorem easily follows from the following claim (using (C6) and (C7)):

Claim. If $\mathscr{D}$ is a *WC*-normal derivation that does not end in Weakening or Contraction, then

1. $\mathscr{D}_1^m, \mathscr{D}_0$ are *WC*-normal derivations that do not end in Weakening or Contraction;
2. $\Gamma_1, \ldots, \Gamma_m = \Gamma$ ($\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$, and $\Gamma_1 \cup \cdots \cup \Gamma_m = \Gamma$);
3. $\Delta_0 = \Delta$;
4. $$\cfrac{\left(\begin{array}{c}\mathscr{D}_i \\ \Gamma_i \Rightarrow S_i : E_i\end{array}\right)_{i=1}^m \quad \mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow P : C}{\Gamma, \Delta \Rightarrow P[(S_i/z_i)_{i=1}^m] : C}\ \mathrm{Cut} \quad \twoheadrightarrow_\flat \mathscr{D}.$$

The claim can be shown by straightforward induction on $\mathscr{D}$. We omit the proof in the interest of space. ◻

Remark. Let $\mathscr{D}$ be a cut-free derivation and let $\tilde{\mathscr{D}}$ be a *WC*-normal form of it. The results of applying Maehara's method to $\mathscr{D}$ and $\tilde{\mathscr{D}}$ may be different.

We note that Theorems 6, 8, and 9 relativize to $R_\rightarrow$, *BCK*-logic, and *BCI*-logic. Conditions 1 and 2 of Definition 5 are strengthened for these substructural logics. For $R_\rightarrow$, they are replaced by

1. $\Gamma_1 \cup \cdots \cup \Gamma_m = \Gamma$;
2. $\Delta_0 = \Delta$.

For *BCK*-logic, the following condition is added to the original conditions 1 and 2:

1^+. $\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$.

For *BCI*-logic, the original conditions are replaced by the combination of the above three conditions, or equivalently:

1. $\Gamma_1, \ldots, \Gamma_m = \Gamma$;
2. $\Delta_0 = \Delta$.

2.3 Which interpolants are computed by Maehara's method?

The same provable sequent has many different cut-free derivations in general. When Maehara's method is applied to two different cut-free derivations of the same labeled sequent $\Gamma, \Delta \Rightarrow T : C$ with respect to the same partition $(\Gamma; \Delta)$, one may get $LJ_{\to}$-interpolants giving different interpolation sequences. The following is a simple example, where we decorate each sequent with a semicolon indicating the partition of the antecedent as well as with the interpolation sequence obtained by Maehara's method at the relevant step.

$$\dfrac{x_1 : p_1 \;;\; \overset{p_1}{\Longrightarrow} x_1 : p_1 \qquad \dfrac{u : p_2 \;;\; \overset{p_2}{\Longrightarrow} u : p_2 \qquad v : p_3 \;;\; \overset{p_3}{\Longrightarrow} v : p_3}{x_2 : p_2 \to p_3 \;;\; u : p_2 \overset{p_2 \to p_3}{\Longrightarrow} x_2 u : p_3}\,(\to\Rightarrow)}{x_1 : p_1,\; x_2 : p_2 \to p_3 \;;\; y_1 : p_1 \to p_2 \overset{p_1, p_2 \to p_3}{\Longrightarrow} x_2(y_1 x_1) : p_3}\,(\to\Rightarrow)$$

$$\dfrac{\dfrac{x_1 : p_1 \;;\; \overset{p_1}{\Longrightarrow} x_1 : p_1 \qquad u : p_2 \;;\; \overset{p_2}{\Longrightarrow} u : p_2}{y_1 : p_1 \to p_2 \;;\; x_1 : p_1 \overset{p_1 \to p_2}{\Longrightarrow} y_1 x_1 : p_2}\,(\to\Rightarrow) \qquad v : p_3 \;;\; \overset{p_3}{\Longrightarrow} v : p_3}{x_1 : p_1,\; x_2 : p_2 \to p_3 \;;\; y_1 : p_1 \to p_2 \overset{(p_1 \to p_2) \to p_3}{\Longrightarrow} x_2(y_1 x_1) : p_3}\,(\to\Rightarrow)$$

Systematically applying Maehara's method to all cut-free derivations of a sequent with a given partition, one can in general find a large number of $LJ_{\to}$-interpolants. One might ask whether all $LJ_{\to}$-interpolants may be found in this way. The answer is negative.

Example 10. Consider the following sequent together with the indicated partition:

(3) $x_1{:}((p_1{\to}p_2){\to}p_5){\to}p_6,\; x_2{:}p_3{\to}p_4,\; x_3{:}p_1 \;;\; y_1{:}p_4{\to}p_5,\; y_2{:}p_2{\to}p_3 \Rightarrow x_1(\lambda u.y_1(x_2(y_2(u x_3)))){:}p_6$

The following *WC*-normal derivations satisfy the conditions 1–5 of Definition 5 (as well as the condition in Theorem 8):

$$\mathscr{D}_1 = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x_3 : p_1 \Rightarrow x_3 : p_1 \qquad u_4 : p_2 \Rightarrow u_4 : p_2}{u_2 : p_1 \to p_2,\; x_3 : p_1 \Rightarrow u_2 x_3 : p_2}\,(\to\Rightarrow) \qquad u_5 : p_5 \Rightarrow u_5 : p_5}{u_2 : p_1 \to p_2,\; u_1 : p_2 \to p_5,\; x_3 : p_1 \Rightarrow u_1(u_2 x_3) : p_5}\,(\to\Rightarrow)}{u_1 : p_2 \to p_5,\; x_3 : p_1 \Rightarrow \lambda u_2.u_1(u_2 x_3) : (p_1 \to p_2) \to p_5}\,(\Rightarrow\to) \qquad u_6 : p_6 \Rightarrow u_6 : p_6}{u_1 : p_2 \to p_5,\; x_1 : ((p_1 \to p_2) \to p_5) \to p_6,\; x_3 : p_1 \Rightarrow x_1(\lambda u_2.u_1(u_2 x_3)) : p_6}\,(\to\Rightarrow)}{x_1 : ((p_1 \to p_2) \to p_5) \to p_6,\; x_3 : p_1 \Rightarrow \lambda u_1.x_1(\lambda u_2.u_1(u_2 x_3)) : (p_2 \to p_5) \to p_6}\,(\Rightarrow\to)$$

$$\mathscr{D}_2 = \; x_2 : p_3 \to p_4 \Rightarrow x_2 : p_3 \to p_4$$

$$\mathscr{D}_0 = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v_1 : p_2 \Rightarrow v_1 : p_2 \qquad v_2 : p_3 \Rightarrow v_2 : p_3}{v_1 : p_2,\; y_2 : p_2 \to p_3 \Rightarrow y_2 v_1 : p_3}\,(\to\Rightarrow) \qquad v_3 : p_4 \Rightarrow v_3 : p_4}{v_1 : p_2,\; z_2 : p_3 \to p_4,\; y_2 : p_2 \to p_3 \Rightarrow z_2(y_2 v_1) : p_4}\,(\to\Rightarrow) \qquad v_4 : p_5 \Rightarrow v_4 : p_5}{v_1 : p_2,\; z_2 : p_3 \to p_4,\; y_1 : p_4 \to p_5,\; y_2 : p_2 \to p_3 \Rightarrow y_1(z_2(y_2 v_1)) : p_5}\,(\to\Rightarrow)}{z_2 : p_3 \to p_4,\; y_1 : p_4 \to p_5,\; y_2 : p_2 \to p_3 \Rightarrow \lambda v_1.y_1(z_2(y_2 v_1)) : p_2 \to p_5}\,(\Rightarrow\to) \qquad v_5 : p_6 \Rightarrow v_5 : p_6}{z_1 : (p_2 \to p_5) \to p_6,\; z_2 : p_3 \to p_4,\; y_1 : p_4 \to p_5,\; y_2 : p_2 \to p_3 \Rightarrow z_1(\lambda v_1.y_1(z_2(y_2 v_1))) : p_6}\,(\to\Rightarrow)$$

However, there is no cut-free derivation of (3) on which Maehara's method returns these derivations. In fact, we can make a stronger claim: there is no cut-free derivation of

(4) $x_1{:}((p_1{\to}p_2){\to}p_5){\to}p_6,\ x_2{:}p_3{\to}p_4,\ x_3{:}p_1;y_1{:}p_4{\to}p_5,\ y_2{:}p_2{\to}p_3 \Rightarrow U{:}p_6,$

for any U, on which Maehara's method returns derivations giving the interpolation sequence:[10]

$$(p_2 \to p_5) \to p_6,\ p_3 \to p_4.$$

Let us call the first part Γ of a partition $(\Gamma;\Delta)$ the *selected part* and the second part Δ the *unselected part*. To see that our claim holds, note that, for Maehara's method to produce a multiple-formula interpolation sequence, a cut-free derivation must have an application of $(\to\Rightarrow)$ that introduces a formula in the unselected part somewhere on the rightmost branch of the derivation. Let $\mathscr{D}$ be a cut-free derivation of (4). By the remark following Theorem 6, we can assume that $\mathscr{D}$ is W-normal. Since λ-terms are immaterial, we omit all λ-term labels and work with unlabeled sequents, treating antecedents of sequents as multisets of formulas. Observe that since none of the following sequents

$$((p_1 \to p_2) \to p_5) \to p_6,\ p_3 \to p_4,\ p_1,\ p_4 \to p_5,\ p_2 \to p_3 \Rightarrow p_3,$$
$$((p_1 \to p_2) \to p_5) \to p_6,\ p_3 \to p_4,\ p_1,\ p_4 \to p_5,\ p_2 \to p_3 \Rightarrow p_4,$$
$$((p_1 \to p_2) \to p_5) \to p_6,\ p_3 \to p_4,\ p_1,\ p_4 \to p_5,\ p_2 \to p_3 \Rightarrow p_2,$$

are even classically valid, the last formula introduced by an operational inference in $\mathscr{D}$ must be $((p_1 \to p_2) \to p_5) \to p_6$. So $\mathscr{D}$ must look like:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{E}_1\\ \Delta_1\ ;\ \Gamma_1 \Rightarrow (p_1 \to p_2) \to p_5\end{matrix} \quad \begin{matrix}\mathscr{E}_2\\ p_6, \Gamma_2\ ;\ \Delta_2 \Rightarrow p_6\end{matrix}}{((p_1 \to p_2) \to p_5) \to p_6, \Gamma_1, \Gamma_2\ ;\ \Delta_1, \Delta_2 \Rightarrow p_6}\,(\to\Rightarrow)}{((p_1 \to p_2) \to p_5) \to p_6,\ p_3 \to p_4,\ p_1\ ;\ p_4 \to p_5,\ p_2 \to p_3 \Rightarrow p_6}\ \text{Contr, Weak}$$

Here, Γ_1 and Γ_2 are multisets consisting of some of the formulas in $((p_1 \to p_2) \to p_5) \to p_6, p_3 \to p_4, p_1$, and Δ_1 and Δ_2 are multisets consisting of some of the formulas in $p_4 \to p_5, p_2 \to p_3$. Since $\mathscr{D}$ is W-normal, $\mathscr{E}_2$ is a W-normal derivation which does not end in Weakening. It follows that p_6 in $\mathscr{E}_2$ cannot have been introduced by Weakening, and $\mathscr{E}_2$ must simply be an initial sequent $p_6 \Rightarrow p_6$ ($\Gamma_2 = \Delta_2 = \varnothing$). We have shown that the only operational inference on the rightmost branch of $\mathscr{D}$ introduces $((p_1 \to p_2) \to p_5) \to p_6$ in the selected part of the partition.

With a slightly more complex example, one can show that Maehara's method sometimes misses $LJ_{\to}$-interpolants of length 1, including those satisfying the additional condition in Theorem 8.

3 Interpolation in Natural Deduction

We define the set of *deductions* in the system $NJ_{\to}$ of natural deduction by induction, simultaneously with two functions: the function $\mathrm{Ass}(\mathscr{D})$ assigning contexts to deductions and the function $\mathrm{Endf}(\mathscr{D})$ assigning formulas to deductions.

[10] By the Coherence Theorem (see Mints 2000), U must be the term in (3).

NJ$_{\to}$.

Assumptions. If x is a variable and A a formula, $\mathscr{D} = x : A$ is a deduction, and $\mathrm{Ass}(\mathscr{D}) = \{x : A\}$ and $\mathrm{Endf}(\mathscr{D}) = A$.

Elimination. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are deductions such that $\mathrm{Endf}(\mathscr{D}_1) = A \to B$, and $\mathrm{Endf}(\mathscr{D}_2) = A$, then

$$\mathscr{D} = \frac{\mathscr{D}_1 \quad \mathscr{D}_2}{B} \to E$$

is a deduction, $\mathrm{Ass}(\mathscr{D}) = \mathrm{Ass}(\mathscr{D}_1) \cup \mathrm{Ass}(\mathscr{D}_2)$, and $\mathrm{Endf}(\mathscr{D}) = B$.

Introduction. If $\mathscr{D}_1$ is a deduction with $\mathrm{Endf}(\mathscr{D}_1) = B$, then

$$\mathscr{D} = \frac{\mathscr{D}_1}{A \to B} \to I, x$$

is a deduction, $\mathrm{Ass}(\mathscr{D}) = \mathrm{Ass}(\mathscr{D}_1) - \{x : A\}$, and $\mathrm{Endf}(\mathscr{D}) = A \to B$.

Each occurrence in $\mathscr{D}$ of $x : A \in \mathrm{Ass}(\mathscr{D})$ is called an *assumption*. Each member $x : A$ of $\mathrm{Ass}(\mathscr{D})$ represents an *assumption class*, namely the set of all assumptions in $\mathscr{D}$ of the form $x : A$. We say that the last inference in

$$\frac{\mathscr{D}_1}{A \to B} \to I, x$$

discharges all assumptions of the form $x : A$ in $\mathscr{D}_1$. If $x : A \notin \mathrm{Ass}(\mathscr{D}_1)$, we say that this inference is a *vacuous* application of $\to I$, and say that the occurrence of A in its conclusion is *introduced* by this inference. We assume that variables in a deduction are so chosen that if a deduction $\mathscr{D}$ has a subdeduction of the above form, $x : A \notin \mathrm{Ass}(\mathscr{D})$.

The occurrence of $\mathrm{Endf}(\mathscr{D})$ at the bottom of $\mathscr{D}$ is called the *endformula* of $\mathscr{D}$.

If $\mathscr{D}$ is a deduction with $\mathrm{Ass}(\mathscr{D}) = \Gamma$, $\mathrm{Endf}(\mathscr{D}) = C$, we write $\mathscr{D} : \Gamma \Rightarrow C$ and often depict $\mathscr{D}$ by

$$\begin{matrix} \Gamma \\ \mathscr{D} \\ C \end{matrix}$$

We follow the same convention on the use of commas in representing contexts as in the case of $LJ_{\to}$.

The $NJ_{\to}$ deductions are in obvious correspondence with λ-terms. We take for granted the notions of *substitution*, *β-redex*, *β-reduction*, *normalization*, and *normal form*. When $\mathscr{D}_1 : \Gamma \Rightarrow A$, $\mathscr{D}_2 : \Delta \Rightarrow B$ and x is a variable of type A, we write $\mathscr{D}_2[\mathscr{D}_1/x]$ for the result of substituting $\mathscr{D}_1$ for x in $\mathscr{D}_2$. We write $\mathscr{D}_1 \twoheadrightarrow_\beta \mathscr{D}_2$ when $\mathscr{D}_1$ β-reduces to $\mathscr{D}_2$. We write $|\mathscr{D}|_\beta$ for the normal form of $\mathscr{D}$, and write $\mathscr{D}_1 =_\beta \mathscr{D}_2$ when $|\mathscr{D}_1|_\beta = |\mathscr{D}_2|_\beta$.

Consider a β-redex

$$\cfrac{\cfrac{\begin{matrix} \vdots\, \mathscr{D}_1 \\ B \end{matrix}}{A \to B} \to I, x \qquad \begin{matrix} \vdots\, \mathscr{D}_2 \\ A \end{matrix}}{B} \to E$$

occurring in some deduction $\mathscr{D}$. We call the exhibited occurrence of $A \to B$ a *redex formula*. Let $\mathscr{D}'$ be the result of contracting this β-redex in $\mathscr{D}$. The β-reduction step from $\mathscr{D}$ to $\mathscr{D}'$ is called *erasing* if $x : A \notin \mathrm{Ass}(\mathscr{D}_1)$; otherwise it is *non-erasing*. If $\mathscr{D}_1$ has more than one assumption of the form $x : A$, then this β-reduction step is called *duplicating*; otherwise *non-duplicating*.

We use the abbreviatory conventions introduced in section 2.2. Moreover, we adopt the following conventions.

$$\begin{array}{c}\Gamma\\ \mathscr{D}_1^n\\ C_1^n\end{array} \quad \text{abbreviates} \quad \begin{pmatrix}\Gamma_i\\ \mathscr{D}_i\\ C_i\end{pmatrix}_{i=1}^{n} \quad \text{if } \Gamma_1 \cup \cdots \cup \Gamma_n = \Gamma.$$

$$\dfrac{\begin{array}{c}\Gamma\\ \mathscr{D}\\ B\end{array}}{\vec{A} \to B}\;{\to}I, \vec{u} \quad \text{abbreviates} \quad \begin{array}{c}\dfrac{\begin{array}{c}\Gamma\\ \mathscr{D}\\ B\end{array}}{A_n \to B}\;{\to}I, u_n\\ \vdots\\ \dfrac{A_2^n \to B}{A_1^n \to B}\;{\to}I, u_1\end{array} \quad \text{if } \vec{A} \text{ represents } A_1^n \text{ and } \vec{u} \text{ represents } u_1^n.$$

We now describe a method of transforming normal natural deductions into cut-free $LJ_{\to}$-derivations, which is essentially the same as the transformation described by Prawitz (1965) (see also Troelstra and Schwichtenberg 2000). If T is a λ-term, let us write $\mathscr{D}_T$ for the $NJ_{\to}$-deduction corresponding to T.

Lemma 11. *Given a normal natural deduction $\mathscr{D}\colon \Gamma \Rightarrow C$, one can construct a cut-free W-normal $LJ_{\to}$-derivation $g(\mathscr{D})\colon \Gamma \Rightarrow P{:}C$ that does not end in Weakening such that $\mathscr{D}_P = \mathscr{D}$.*

Proof. By induction on the height of $\mathscr{D}$.

Induction Basis. $\mathscr{D}$ is an assumption $x : C$. Let $g(\mathscr{D}) = x : C \Rightarrow x : C$.

Induction Step.

Case 1. The last inference of $\mathscr{D}$ is $\to I$. $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}(x : A)^{\circ}, \Gamma\\ \mathscr{D}'\\ B\end{array}}{A \to B}\;{\to}I, x$$

Case 1.1. $x : A \in \mathrm{Ass}(\mathscr{D}')$. By the induction hypothesis, we have an $LJ_{\to}$-derivation $g(\mathscr{D}')\colon x : A, \Gamma \Rightarrow T : B$. Let

$$g(\mathscr{D}) = \dfrac{\begin{array}{c}g(\mathscr{D}')\\ x : A, \Gamma \Rightarrow T : B\end{array}}{\Gamma \Rightarrow \lambda x.T : A \to B}\;(\Rightarrow\to)$$

Case 1.2. $x : A \notin \mathrm{Ass}(\mathscr{D}')$. By the induction hypothesis, we have an $LJ_{\to}$-derivation $g(\mathscr{D}')\colon \Gamma \Rightarrow T : B$. Let

$$g(\mathscr{D}) = \dfrac{\dfrac{\begin{array}{c}g(\mathscr{D}')\\ \Gamma \Rightarrow T : B\end{array}}{x : A, \Gamma \Rightarrow T : B}\;\text{Weak}}{\Gamma \Rightarrow \lambda x.T : A \to B}\;(\Rightarrow\to)$$

Case 2. The last inference of $\mathscr{D}$ is $\to E$. We analyze $\mathscr{D}$ as follows, tracing its main branch:

$$\dfrac{\dfrac{x : C_1^k \to C \quad \begin{array}{c}\Gamma'\\ \mathscr{D}'\\ C_1\end{array}}{C_2^k \to C}\;{\to}E \qquad \begin{array}{c}\Gamma''\\ \mathscr{E}_2^k\\ C_2^k\end{array}}{C}\;{\to}E$$

where $\{x : C_1^k \to C\} \cup \Gamma' \cup \Gamma'' = \Gamma$. Letting $x\mathscr{D}'$ denote the deduction

$$\dfrac{x : C_1^k \to C \quad \begin{array}{c}\Gamma'\\ \mathscr{D}'\\ C_1\end{array}}{C_2^k \to C} \to E$$

and $\mathscr{D}''$ denote

$$\dfrac{y : C_2^k \to C \quad \begin{array}{c}\Gamma''\\ \mathscr{E}_2^k\\ C_2^k\end{array}}{C} \to E$$

where y is a fresh variable, we can write

$$\mathscr{D} = \mathscr{D}''[x\mathscr{D}'/y].$$

By the induction hypothesis, we have $LJ_\to$-derivations $g(\mathscr{D}') \colon \Gamma' \Rightarrow U : C_1$ and $g(\mathscr{D}'') \colon y : C_2^k \to C, \Gamma'' \Rightarrow T : C$.

Case 2.1. $x : C_1^k \to C \notin \Gamma' \cup \Gamma''$. Let

$$g(\mathscr{D}) = \dfrac{\begin{array}{c} g(\mathscr{D}') \\ \Gamma' \Rightarrow U : C_1 \end{array} \quad \begin{array}{c} g(\mathscr{D}'') \\ y : C_2^k \to C, \Gamma'' \Rightarrow T : C \end{array}}{x : C_1^k \to C, \Gamma' \cup \Gamma'' \Rightarrow T[xU/y] : C} \, (\to\Rightarrow)\dagger$$

Case 2.2. $x : C_1^k \to C \in \Gamma' \cup \Gamma''$. We write Γ_0 for $\Gamma' \cup \Gamma'' - \{x : C_1^k \to C\}$. Let w be a fresh variable, and let

$$g(\mathscr{D}) = \dfrac{\dfrac{\begin{array}{c} g(\mathscr{D}')[w/x] \\ \Gamma'[w/x] \Rightarrow U[w/x] : C_1 \end{array} \quad \begin{array}{c} g(\mathscr{D}'')[w/x] \\ y : C_2^k \to C, \Gamma''[w/x] \Rightarrow T[w/x] : C \end{array}}{z : C_1^k \to C, w : C_1^k \to C, \Gamma_0 \Rightarrow T[w/x][zU[w/x]/y] : C} \, (\to\Rightarrow)\dagger}{x : C_1^k \to C, \Gamma_0 \Rightarrow T[xU/y] : C} \, \text{Contr}$$

This completes the construction of $g(\mathscr{D})$. It is easy to check that $g(\mathscr{D})$ satisfies the required properties in all cases. □

3.1 Links in natural deduction

As in $LJ_\to$, we associate with each occurrence of a propositional variable in a natural deduction two *ports*, which we call the *top port* and the *bottom port*. We decorate natural deductions with *links* connecting two ports inductively as follows (p stands for an arbitrary propositional variable):

Elimination rule.

$$\dfrac{A[p] \to B[p] \quad A[p]}{B[p]} \to E$$

We draw a link between

- the top port of an occurrence of p in B in the conclusion and the bottom port of the corresponding occurrence of p in B in the major premise;

- the bottom port of an occurrence of p in A in the major premise and the bottom port of the corresponding occurrence of p in A in the minor premise.

Introduction rule.

$$\dfrac{\begin{matrix} x : A[p] \\ \vdots \\ B[p] \end{matrix}}{A[p] \to B[p]} \;{\to}I, x$$

We draw a link between

- the top port of an occurrence of p in B in the conclusion of this inference and the bottom port of the corresponding occurrence of p in B immediately above;

- the top port of an occurrence of p in A in the conclusion of this inference and the top port of the corresponding occurrence of p in each of the assumptions $x : A$ that are discharged by this inference.

The notions of *path*, *maximal path*, and *cycle* are understood in exactly the same way as in the case of sequent calculus. Two occurrences of a propositional variable in a natural deduction are *linked* to each other if there is a path that starts in one and ends in the other. The relation of *being linked to* is reflexive and symmetric, but not transitive. Our definition of links is slightly different from the similar definition of *connection* in Hirokawa 1993. A maximal path starts and ends either in an occurrence inside an assumption, an occurrence inside the endformula, or an occurrence inside a formula occurrence introduced by a vacuous application of ${\to}I$. In a normal deduction, at least one of the endpoints of a maximal path must be of one of the first two types. It is not difficult to see that no cycle can occur in a normal deduction.

Example 12. In the following natural deduction, we refer to different occurrences of p by the numbers attached to them:

$$\dfrac{\dfrac{\dfrac{x : (\overset{[1]}{p} \to q) \to (\overset{[2]}{p} \to q) \to p \qquad y : p \to q}{(p \to q) \to p}\;{\to}E \qquad y : p \to q}{p}\;{\to}E}{(\underset{[3]}{p} \to q) \to p}\;{\to}I, y$$

In this deduction, both [1] and [2] are linked to [3], but [1] is *not* linked to [2].

If $\mathscr{D} \colon \Gamma \Rightarrow T : C$ is a cut-free $LJ_{\to}$-derivation, then we write $n(\mathscr{D})$ for the normal natural deduction $\mathscr{D}_T$.

Lemma 13. *Let $\mathscr{D} \colon \Gamma \Rightarrow T : C$ be a cut-free W-normal $LJ_{\to}$-derivation that does not end in Weakening. Then $n(\mathscr{D}) \colon \Gamma \Rightarrow C$, and there is an onto function f from the maximal paths in $n(\mathscr{D})$ to the maximal paths in $\mathscr{D}$ such that for every maximal path π in $n(\mathscr{D})$:*

1. *if π starts (ends) in an occurrence of p in a formula occurrence $A[p]$ introduced by a vacuous application of $\to I$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in a formula occurrence $A[p]$ introduced by an application of Weakening;*

2. *if π starts (ends) in an occurrence of p in an assumption $x : A[p]$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in $x : A[p]$ in the antecedent of the endsequent;*

3. *if π starts (ends) in an occurrence of p in the endformula $C[p]$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in the succedent $C[p]$ of the endsequent.*

Let us call an occurrence of a propositional variable that appears inside a redex formula a *redex-internal occurrence*.

Lemma 14. *Suppose that an $NJ_{\to}$-deduction $\mathscr{D} : \Gamma \Rightarrow C$ reduces to $\mathscr{D}' : \Gamma \Rightarrow C$ by a sequence of non-erasing β-reduction steps. Then there is a function f from the set of maximal paths in $\mathscr{D}'$ to the set of maximal paths in $\mathscr{D}$ such that for every maximal path π in $\mathscr{D}'$:*

1. *if π starts (ends) in an occurrence of p in a formula occurrence $A[p]$ introduced by a vacuous application of $\to I$ in $\mathscr{D}'$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in a formula occurrence $A[p]$ introduced by a vacuous application of $\to I$ in $\mathscr{D}$;*

2. *if π starts (ends) in an occurrence of p in an assumption $x : A[p]$ of $\mathscr{D}'$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in an assumption $x : A[p]$ of $\mathscr{D}$;*

3. *if π starts (ends) in an occurrence of p in the endformula $C[p]$ of $\mathscr{D}'$, then $f(\pi)$ starts (ends) in the corresponding occurrence of p in the endformula $C[p]$ of $\mathscr{D}$;*

4. *if π contains k redex-internal occurrences, then $f(\pi)$ contains at least k redex-internal occurrences.*

Proof. Clearly, it suffices to consider the case of one-step β-reduction. Suppose that $\mathscr{D}$ reduces to $\mathscr{D}'$ in one non-erasing β-reduction step. We can depict $\mathscr{D}$ and $\mathscr{D}'$ as follows:

(5)

$$\mathscr{D} = \begin{array}{c} \ldots\quad x:A \quad \ldots \quad x:A \quad \ldots \\ \vdots\,\mathscr{D}_1 \\ \dfrac{\dfrac{B}{A \to B}\,{\to}I, x \qquad \begin{array}{c} \vdots\,\mathscr{D}_2 \\ A \end{array}}{B}\,{\to}E \\ \vdots \end{array} \qquad \mathscr{D}' = \begin{array}{c} \begin{array}{c} \vdots\,\mathscr{D}_2 \\ \ldots\quad A \end{array} \quad \ldots \quad \begin{array}{c} \vdots\,\mathscr{D}_2 \\ A \quad \ldots \end{array} \\ \vdots\,\mathscr{D}_1 \\ B \\ \vdots \end{array}$$

In general, $\mathscr{D}_1$ has $n \geq 1$ occurrences of $x : A$; the above figure represents the case where $n = 2$.

We can map each occurrence o' of a propositional variable and its top and bottom ports ρ'_t, ρ'_b in the dotted parts of $\mathscr{D}'$ (i.e., those that are not inside the exhibited occurrences of A and B) to the corresponding occurrence and ports o, ρ_t, ρ_b in the dotted parts of $\mathscr{D}$ in an obvious way. As for the remaining propositional variable occurrences and their ports, we consider two cases.

Case 1. $\mathscr{D}_1$ is an assumption $x : A$. Then the situation looks like

$$\mathscr{D} = \dfrac{\dfrac{x : A[p^{[3]}]}{A[p^{[2]}] \to A[p^{[4]}]} \to I, x \qquad \begin{matrix} \vdots\, \mathscr{D}_2 \\ A[p^{[1]}] \end{matrix}}{\begin{matrix} A[p^{[5]}] \\ \vdots \end{matrix}} \to E \qquad \mathscr{D}' = \begin{matrix} \vdots\, \mathscr{D}_2 \\ A[p] \\ \vdots \end{matrix}$$

Consider an occurrence o' of p inside the exhibited occurrence of A in $\mathscr{D}'$. Let ρ'_t and ρ'_b be its top port and bottom port, respectively. We map a subpath (ρ'_t, o', ρ'_b) of a maximal path in $\mathscr{D}'$ to

$$(\rho_t^{[1]}, [1], \rho_b^{[1]}, \rho_b^{[2]}, [2], \rho_t^{[2]}, \rho_t^{[3]}, [3], \rho_b^{[3]}, \rho_t^{[4]}, [4], \rho_b^{[4]}, \rho^{[5]}, [5], \rho_b^{[5]})$$

where $\rho_t^{[i]}, \rho_b^{[i]}$ are the top and bottom ports of the occurrence of p indicated by $[i]$ in the above figure. The reverse subpath (ρ'_b, o', ρ'_t) is mapped to the reverse of the above sequence.

Case 2. $\mathscr{D}_1$ ends in $\to E$ or $\to I$. In this case, the exhibited occurrences of A and B in (5) are all distinct. Let us consider an occurrence o' of p inside the i-th exhibited occurrence of A in $\mathscr{D}'$. We depict the case where $i = 2$:

$$\mathscr{D} = \quad \begin{matrix} \dots \quad x : A \quad \dots \quad x : A[p^{[3]}] \quad \dots \\ \vdots\, \mathscr{D}_1 \\ \dfrac{\dfrac{B[p^{[4]}]}{A[p^{[2]}] \to B[p^{[5]}]} \to I, x \qquad \begin{matrix} \vdots\, \mathscr{D}_2 \\ A[p^{[1]}] \end{matrix}}{\begin{matrix} B[p^{[6]}] \\ \vdots \end{matrix}} \to E \end{matrix} \qquad \mathscr{D}' = \begin{matrix} \begin{matrix} \vdots\, \mathscr{D}_2 \\ \dots \quad A \quad \dots \end{matrix} \quad \begin{matrix} \vdots\, \mathscr{D}_2 \\ A[p] \quad \dots \end{matrix} \\ \vdots\, \mathscr{D}_1 \\ B[p] \\ \vdots \end{matrix}$$

Let ρ'_t and ρ'_b be the top and bottom port of o', respectively. We write $\rho_t^{[i]}$ and $\rho_b^{[i]}$ for the top and bottom port of the occurrence of p indicated by $[i]$ in the above figure. We map a subpath (ρ'_t, o', ρ'_b) to

$$(\rho_t^{[1]}, [1], \rho_b^{[1]}, \rho_b^{[2]}, [2], \rho_t^{[2]}, \rho_t^{[3]}, [3], \rho_b^{[3]}).$$

The reverse subpath (ρ'_b, o', ρ'_t) is mapped to the reverse sequence. Now let us consider an occurrence o' of p inside the exhibited occurrence of B in $\mathscr{D}'$. Let ρ'_t and ρ'_b be its top and bottom port, respectively. We map a subpath (ρ'_t, o', ρ'_b) to

$$(\rho_t^{[4]}, [4], \rho_b^{[4]}, \rho_t^{[5]}, [5], \rho_b^{[5]}, \rho_t^{[6]}, [6], \rho_b^{[6]}).$$

The reverse subpath (ρ'_b, o', ρ'_t) is mapped to the reverse sequence.

We have described a way of mapping every maximal path π in $\mathscr{D}'$ to a sequence $f(\pi)$. It is not difficult to see that $f(\pi)$ is a maximal path in $\mathscr{D}$ and satisfies the requirements of the lemma. We leave the details to the reader. □

The function f in Lemma 14 need not be onto. For example, consider the following deduction $\mathscr{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : q \to r \to s \qquad \dfrac{y : (p \to p) \to q \qquad u : p \to p}{q}\,{\to}E}{r \to s}\,{\to}E \qquad \dfrac{z : (p \to p) \to r \qquad u : p \to p}{r}\,{\to}E}{s}\,{\to}E}{(p \to p) \to s}\,{\to}I, u \qquad \dfrac{v : p}{p \to p}\,{\to}I, v}{s}\,{\to}E$$

There is a maximal path starting in the first occurrence of p in $y : (p \to p) \to q$ and ending in the second occurrence of p in $z : (p \to p) \to r$ in $\mathscr{D}$, but there is no corresponding path in $|\mathscr{D}|_\beta$:

$$\dfrac{\dfrac{x : q \to r \to s \qquad \dfrac{y : (p \to p) \to q \qquad \dfrac{v : p}{p \to p}\,{\to}I, v}{q}\,{\to}E}{r \to s}\,{\to}E \qquad \dfrac{z : (p \to p) \to r \qquad \dfrac{v : p}{p \to p}\,{\to}I, v}{r}\,{\to}E}{s}\,{\to}E$$

We can show the following:

Lemma 15. *Let $\mathscr{D}$ and $\mathscr{D}'$ be as in Lemma 14. If $\mathscr{D}$ has no non-trivial path that starts and ends inside the same redex formula, then there is an onto function f from the set of maximal paths in $\mathscr{D}'$ to the set of maximal paths in $\mathscr{D}$ that satisfies the conditions 1–4 in Lemma 14.*

Proof. By part 4 of Lemma 14, it suffices to prove the lemma in the case of one-step non-erasing β-reduction. We sketch a proof that the function f described in the proof of Lemma 14 is onto. Suppose that π is a maximal path in $\mathscr{D}$. There are two cases to consider.

Case 1. π does not contain any occurrence inside the exhibited occurrences of $x : A$ on the left-hand side of (5). Then π does not contain any occurrences inside the exhibited occurrences of A. The construction of f matches a subpath of π that does not go inside the dotted part of $\mathscr{D}_2$ with a unique path in $\mathscr{D}'$. By matching subpaths of π that are inside the dotted part of $\mathscr{D}_2$ with corresponding subpaths in the dotted part of the first copy of $\mathscr{D}_2$ in $\mathscr{D}'$, one can form a maximal path π' of $\mathscr{D}'$ such that $f(\pi') = \pi$.

Case 2. π contains an occurrence inside the exhibited occurrences of $x : A$ on the left-hand side of (5). Then π contains just one such occurrence, by assumption. Suppose that it is inside the i-th exhibited occurrence of $x : A$. According to the construction of f, any subpath of π that goes neither inside the dotted part of $\mathscr{D}_2$ nor inside the exhibited occurrences of A is matched with a unique path in $\mathscr{D}'$.

Case 2.1. Case 1 of the proof of Lemma 14 holds. By matching subpaths of π that are either wholly inside the dotted part of $\mathscr{D}_2$ or wholly inside the exhibited occurrences of A with corresponding paths in $\mathscr{D}_2$ in $\mathscr{D}'$, one can form a maximal path π' of $\mathscr{D}'$ such that $f(\pi') = \pi$.

Case 2.2. Case 2 of the proof of Lemma 14 holds. By matching subpaths of π that are either wholly inside the dotted part of $\mathscr{D}_2$ or wholly inside the exhibited occurrences of A with corresponding paths in the i-th copy of $\mathscr{D}_2$ in $\mathscr{D}'$, one can form a maximal path π' of $\mathscr{D}'$ such that $f(\pi') = \pi$. □

3.2 Interpolants

Let us say that an assumption of $\mathscr{D}$ *belongs to* $\Gamma \subseteq \mathrm{Ass}(\mathscr{D})$ if it belongs to some assumption class in Γ.

Definition 16. Let $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ be a normal deduction. A sequence of normal deductions $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ is an *interpolant to $\mathscr{D}$ with respect to the partition* $(\Gamma; \Delta)$ if and only if there is a normal deduction $\mathscr{D}_0 \colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ such that

(I1) $\Gamma_1 \cup \cdots \cup \Gamma_m = \Gamma$;

(I2) $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m] \twoheadrightarrow_\beta \mathscr{D}$;

(I3) In $\mathscr{D}_i$ ($i = 1, \dots, m$), every maximal path that starts inside the endformula E_i ends inside an assumption;

(I4) In $\mathscr{D}_0$, every maximal path that starts inside an assumption $z_i : E_i$ ends inside the endformula C or inside an assumption belonging to Δ.

We call the deduction $\mathscr{D}_0$ an *auxiliary deduction* for $\mathscr{D}_1^m, \mathscr{D}$, and say that $\mathscr{D}_1^m$ is an interpolant to $\mathscr{D}$ (with respect to the partition $(\Gamma; \Delta)$) *via* $\mathscr{D}_0$.

Remark. We can replace condition (I1) of Definition 16 by a weaker one, namely "$\Gamma_i \subseteq \Gamma$ for each i", without changing the notion of interpolant. This is because the weaker condition together with condition (I2) implies condition (I1).

The following is a natural deduction version of Lemma 7.

Lemma 17. *If* $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ *is an interpolant to* $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ *with respect to the partition* $(\Gamma; \Delta)$, *then* $E_1, \dots, E_m$ *is an interpolation sequence for* $\Gamma^-, \Delta^- \Rightarrow C$ *with respect to the partition* $(\Gamma^-; \Delta^-)$.

The converse of Lemma 17 does not hold; see section 4 for an example.

In general, an interpolant may have more than one auxiliary deduction.

Example 18. Let

$$\mathscr{D} = \dfrac{x_1 : q \to r \qquad \dfrac{\dfrac{\dfrac{y : p \to p \to p \to q \quad x_2 : p}{p \to p \to q}{\to}E \qquad x_2 : p}{p \to q}{\to}E \qquad x_2 : p}{q}{\to}E}{r}{\to}E$$

$$\mathscr{D}_1 = \dfrac{\dfrac{x_1 : q \to r \qquad \dfrac{\dfrac{u : p \to p \to q \quad x_2 : p}{p \to q}{\to}E \qquad x_2 : p}{q}{\to}E}{r}{\to}E}{(p \to p \to q) \to r}{\to}I, u$$

Then $\mathscr{D}_1$ is an interpolant to $\mathscr{D}$ with respect to $(x_1 : q \to r, x_2 : p; y : p \to p \to p \to q)$ via

$$\dfrac{z_1 : (p \to p \to q) \to r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : p \to p \to p \to q \quad u : p}{p \to p \to q}{\to}E \qquad u : p}{p \to q}{\to}E \qquad v : p}{q}{\to}E}{p \to q}{\to}I, v}{p \to p \to q}{\to}I, u}{r}{\to}E$$

as well as via

$$\dfrac{z_1 : (p \to p \to q) \to r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : p \to p \to p \to q \quad u : p}{p \to p \to q}{\to}E \qquad v : p}{p \to q}{\to}E \qquad v : p}{q}{\to}E}{p \to q}{\to}I, v}{p \to p \to q}{\to}I, u}{r}{\to}E$$

For a different type of example, see Example 42.

Let $\mathscr{D}\colon \Gamma \Rightarrow T : C$ be a cut-free $LJ_{\to}$-derivation. Clearly, Lemma 13 implies that if $(\mathscr{D}_i\colon \Gamma_i \Rightarrow S_i : E_i)_{i=1}^m$ is an $LJ_{\to}$-interpolant to $\Gamma, \Delta \Rightarrow T : C$ with respect to the partition $(\Gamma; \Delta)$ via an auxiliary derivation $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta_0 \Rightarrow P : C$, then $(n(\mathscr{D}_i))_{i=1}^m$ is an interpolant to $n(\mathscr{D})\colon \Gamma_1 \cup \cdots \cup \Gamma_m, \Delta_0 \Rightarrow C$ with respect to the partition $(\Gamma_1 \cup \cdots \cup \Gamma_m; \Delta_0)$.[11] Thus, one can read off an interpolant from the output of Maehara's method.

Let $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$ be a normal deduction and T be the λ-term corresponding to it. If a sequence of normal deductions $\mathscr{D}_1^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma; \Delta)$, then $(g(\mathscr{D}))_{i=1}^m$ is an $LJ_{\to}$-interpolant to $\Gamma, \Delta \Rightarrow T : C$ with respect to the same partition. This is another easy consequence of Lemma 13.

We now state some general properties of interpolants.

Lemma 19. *Suppose that* $(\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ *is an interpolant to* $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$ *with respect to the partition* $(\Gamma; \Delta)$ *via an auxiliary deduction* $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$. *Then every reduction sequence from* $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m]$ *to* $\mathscr{D}$ *consists entirely of non-erasing* β*-reduction steps.*

Proof. We claim that if $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m] = \mathscr{C}_0 \twoheadrightarrow_\beta \mathscr{C}$, then $\mathscr{C}$ has no redex-internal occurrence that is linked to an occurrence in a formula occurrence introduced by a vacuous application of $\to I$. Then no erasing β-reduction can be applied to $\mathscr{C}$, and the lemma follows. We prove our claim by induction on the number of β-reduction steps. It is clear that $\mathscr{C}_0$ satisfies the required condition by the definition of an interpolant. Now assume that $\mathscr{C}_i$ satisfies the condition and $\mathscr{C}_i$ β-reduces to $\mathscr{C}_{i+1}$ in one step. This β-reduction step must be non-erasing, so let f be the function from the set of maximal paths in $\mathscr{C}_{i+1}$ to the set of maximal paths in $\mathscr{C}_i$ as described in the proof of Lemma 14. Let π be a maximal path in $\mathscr{C}_{i+1}$ that starts or ends in an occurrence in a formula occurrence introduced by a vacuous application of $\to I$. Then, by condition 1 of Lemma 14, $f(\pi)$ is a maximal path in $\mathscr{C}_i$ that starts or ends in an occurrence in a formula occurrence introduced by a vacuous application of $\to I$. Since $\mathscr{C}_i$ satisfies the condition, $f(\pi)$ contains no redex-internal occurrence. Then by condition 4 of Lemma 14, π cannot contain any redex-internal occurrence, either. Therefore, $\mathscr{C}_{i+1}$ also satisfies the condition. □

The following is an easy consequence of Lemma 19. Let $\#\mathscr{D}$ denote the number of assumptions in $\mathscr{D}$.

Lemma 20. *Let* $\mathscr{D}, \mathscr{D}_1^m, \mathscr{D}_0$ *be as in Lemma 19. Then* $\#\mathscr{D}_1 + \cdots + \#\mathscr{D}_m + \#\mathscr{D}_0 - m \leq \#\mathscr{D}$.

Lemma 21. *Let* $\mathscr{D}, \mathscr{D}_1^m, \mathscr{D}_0$ *be as in Lemma 19. Then there is a function from the set of maximal paths in* $\mathscr{D}$ *onto the set of maximal paths in* $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m]$ *that satisfies the conditions 1–4 in Lemma 14.*

Proof. By the definition of an interpolant, $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m]$ has no path that contains more than one redex-internal occurrence. Since, by Lemma 19, any reduction sequence from $\mathscr{D}_0[(\mathscr{D}_i/z_i)_{i=1}^m]$ to $\mathscr{D}$ consists entirely of non-erasing β-reduction steps, the lemma follows from Lemma 15. □

[11] This will not hold if we drop the requirement of W-normality from the definition of $LJ_{\to}$-interpolant.

Lemma 22. *Let $\mathscr{D}_1^m$ be an interpolant to $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$ with respect to the partition* $(\Gamma; \Delta)$ *via $\mathscr{D}_0$. Then the combined size of $\mathscr{D}_1^m, \mathscr{D}_0$ is bounded by a computable function of the size of $\mathscr{D}$.*[12]

Proof. Since a maximal path in a normal deduction cannot contain a cycle, Lemma 21 implies that the number of assumptions discharged by a single application of $\to I$ in $\mathscr{D}_1^m, \mathscr{D}_0$ is bounded by the number of maximal paths in $\mathscr{D}$. The lemma then follows from a result by Dougherty and Wierzbicki (2002). ◻

Theorem 23. *The problem of determining whether $\mathscr{D}_1^m$ is an interpolant to $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$ with respect to the partition* $(\Gamma; \Delta)$ *is decidable.*

3.3 Prawitz's method

As we did with Maehara's method, we reformulate Prawitz's (1965) method for the implicational fragment using sequences of formulas in place of formulas. We shall see that Prawitz's interpolant is just one of the interpolants found by Maehara's method, so it gives nothing new.

Theorem 24. *Given a normal deduction $\mathscr{D}\colon \Gamma, \Delta \Rightarrow C$, one can find an interpolant to $\mathscr{D}$ with respect to the partition* $(\Gamma; \Delta)$.

Proof. By induction on $\mathscr{D}$. At each step we construct $\mathscr{D}_1^m, \mathscr{D}_0$ satisfying the conditions (I1)–(I4) of Definition 16. We also prove that these deductions satisfy the additional condition:

(*) If the main branch of $\mathscr{D}$ leads to an assumption belonging to Γ, then $m = 1$.

Induction Basis. $\mathscr{D}$ is $x : C$. *Case 1.* $\Gamma = \{x : C\}, \Delta = \varnothing$. Take $m = 1$ and let $\mathscr{D}_1$ be $\mathscr{D}$, and $\mathscr{D}_0$ be $z_1 : C$, where z_1 is a fresh variable. *Case 2.* $\Gamma = \varnothing, \Delta = \{x : C\}$. Take $m = 0$ and let $\mathscr{D}_0$ be $\mathscr{D}$.

Induction Step.

Case 1. The last inference of $\mathscr{D}$ is $\to I$. $\mathscr{D}$ is of the form:

$$\dfrac{\begin{array}{c}(y : A)^\circ, \Gamma, \Delta \\ \mathscr{D}' \\ B\end{array}}{A \to B}\ \to I, y$$

where $A \to B = C$. Apply the induction hypothesis to $\mathscr{D}'\colon (y : A)^\circ, \Gamma, \Delta \Rightarrow B$ with respect to the partition $(\Gamma; (y : A)^\circ, \Delta)$, and obtain normal deductions $(\mathscr{D}_i'\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ and $\mathscr{D}_0'\colon (z_i : E_i)_{i=1}^m, (y : A)^\circ, \Delta \Rightarrow B$ with the required properties. Let $\mathscr{D}_i$ be $\mathscr{D}_i'$ $(i = 1, \ldots, m)$, and let $\mathscr{D}_0$ be the following deduction:

$$\dfrac{\begin{array}{c}(z_i : E_i)_{i=1}^m, (y : A)^\circ, \Delta \\ \mathscr{D}_0' \\ B\end{array}}{A \to B}\ \to I, y$$

It is easy to see that $\mathscr{D}_1^m, \mathscr{D}_0$ satisfy the required properties.

[12] Note that, in general, there is no bound on the size of $\mathscr{D}'$ such that $\mathscr{D}'$ reduces to $\mathscr{D}$ by a sequence of non-erasing β-reduction steps.

Case 2. The last inference of $\mathscr{D}$ is $\to E$. We analyze $\mathscr{D}$ as in the proof of Lemma 11:

$$\dfrac{\dfrac{x : C_1^k \to C \qquad \begin{matrix}\Gamma', \Delta' \\ \mathscr{D}' \\ C_1\end{matrix}}{C_2^k \to C}\to E \qquad \begin{matrix}\Gamma'', \Delta'' \\ \mathscr{E}_2^k \\ C_2^k\end{matrix}}{C}\to E$$

Here, $\Gamma' \cup \Gamma'' \subseteq \Gamma$ and $\Delta' \cup \Delta'' \subseteq \Delta$. Letting $x\mathscr{D}'$ denote the deduction

$$\dfrac{x : C_1^k \to C \qquad \begin{matrix}\Gamma', \Delta' \\ \mathscr{D}' \\ C_1\end{matrix}}{C_2^k \to C}\to E$$

and $\mathscr{D}''$ denote

$$\dfrac{y : C_2^k \to C \qquad \begin{matrix}\Gamma'', \Delta'' \\ \mathscr{E}_2^k \\ C_2^k\end{matrix}}{C}\to E$$

where y is a fresh variable, we can write

$$\mathscr{D} = \mathscr{D}''[x\mathscr{D}'/y].$$

There are two subcases as to whether $x : C_1^k \to C$ is in Γ or in Δ.

Case 2.1. $x{:}C_1^k{\to}C$ is in Δ. We apply the induction hypothesis to $\mathscr{D}' : \Gamma', \Delta' \Rightarrow C_1$ with respect to the partition $(\Gamma'; \Delta')$, and to $\mathscr{D}'' : y : C_2^k \to C, \Gamma'', \Delta'' \Rightarrow C$ with respect to the partition $(\Gamma''; y : C_2^k \to C, \Delta'')$, and obtain normal deductions with the required properties:

$$(\mathscr{D}_i' : \Gamma_i' \Rightarrow F_i)_{i=1}^n, \quad \mathscr{D}_0' : (w_i : F_i)_{i=1}^n, \Delta' \Rightarrow C_1$$
$$(\mathscr{D}_i'' : \Gamma_i'' \Rightarrow G_i)_{i=1}^p, \quad \mathscr{D}_0'' : (v_i : G_i)_{i=1}^p, y : C_2^k \to C, \Delta'' \Rightarrow C$$

We assume that variables have been chosen in such a way that w_1^n and v_1^p are pairwise distinct. Let $m = n + p$, and let

$$\mathscr{D}_1^m = \mathscr{D}'^n_1, \mathscr{D}''^p_1.$$

We let $\mathscr{D}_0 : (w_i : F_i)_{i=1}^n, (v_i : G_i)_{i=1}^p, \{x : C_1^k \to C\} \cup \Delta' \cup \Delta'' \Rightarrow C$ be $\mathscr{D}_0''[x\mathscr{D}_0'/y]$, where $x\mathscr{D}_0'$ is

$$\dfrac{x : C_1^k \to C \qquad \begin{matrix}(w_i : F_i)_{i=1}^n, \Delta' \\ \mathscr{D}_0' \\ C_1\end{matrix}}{C_2^k \to C}\to E$$

Now we check that $\mathscr{D}_1^m$ and $\mathscr{D}_0$ satisfy conditions (I1)–(I4) of Definition 16. We can easily show conditions (I1) and (I2) using the induction hypothesis. Clearly, $\mathscr{D}_1^m$ satisfy condition (I3) by induction hypothesis. To show that $\mathscr{D}_0$ satisfies condition (I4), consider an arbitrary maximal path π in $\mathscr{D}_0$ that starts inside $w_i : F_i$ or $v_i : G_i$. If π starts inside $w_i : F_i$, it starts inside $\mathscr{D}_0'$. By induction hypothesis, π must end inside some assumption in Δ' or exit $\mathscr{D}_0'$ through C_1. If the latter, π ends inside $x : C_1^k \to C$. Now suppose that π starts inside $v_i : G_i$. By induction hypothesis, π

either ends inside some assumption in Δ'', ends inside the endformula C, or exit $\mathscr{D}_0''$ through $C_2^k \to C$, entering $x\mathscr{D}_0'$. If the last case obtains, π ends inside $x : C_1^k \to C$. We have shown that $\mathscr{D}_0$ satisfies condition (I4).

Case 2.2. $x : C_1^k \to C$ is in Γ. Apply the induction hypothesis to $\mathscr{D}' : \Gamma', \Delta' \Rightarrow C_1$ with respect to the partition $(\Delta'; \Gamma')$, and to $\mathscr{D}'' : y : C_2^k \to C, \Gamma'', \Delta'' \Rightarrow C$ with respect to the partition $(y : C_2 \to C, \Gamma''; \Delta'')$, and obtain normal deductions with the required properties:

$$(\mathscr{D}_i' : \Delta_i' \Rightarrow F_i)_{i=1}^n, \quad \mathscr{D}_0' : (w_i : F_i)_{i=1}^n, \Gamma' \Rightarrow C_1$$
$$\mathscr{D}_1'' : y : C_2^k \to C, \Gamma'' \Rightarrow G_1, \quad \mathscr{D}_0'' : v_1 : G_1, \Delta'' \Rightarrow C$$

where v_1 is distinct from any of w_1^n. Note that the main branch of $\mathscr{D}''$ leads to $y : C_2^k \to C$, so the additional condition (*) is satisfied. Let $x\mathscr{D}_0'$ be the deduction

$$\dfrac{x : C_1^k \to C \qquad \begin{array}{c}(w_i : F_i)_{i=1}^n, \Gamma' \\ \mathscr{D}_0' \\ C_1\end{array}}{C_2^k \to C} \to E$$

Let $m = 1$ and let $\mathscr{D}_1 : \{x : C_1 \to C_2\} \cup \Gamma' \cup \Gamma'' \Rightarrow F_1^n \to G_1$ be the following deduction

$$\begin{array}{c}(w_i : F_i)_{i=1}^n, \{x : C_1^k \to C\} \cup \Gamma' \cup \Gamma'' \\ \mathscr{D}_1''[x\mathscr{D}_0'/y] \\ \dfrac{G_1}{\overline{F_1^n \to G_1}} \to I, w_1^n\end{array}$$

and let $\mathscr{D}_0 : z_1 : F_1^n \to G_1, \Delta' \cup \Delta'' \Rightarrow C$ be $\mathscr{D}_0''[z_1\mathscr{D'}_1^n/v_1]$, where $z_1\mathscr{D'}_1^n$ is the following normal deduction:

$$\dfrac{z_1 : F_1^n \to G_1 \qquad \begin{array}{c}\Delta' \\ \mathscr{D'}_1^n \\ F_1^n\end{array}}{\overline{G_1}} \to E$$

We leave to the reader the proof that $\mathscr{D}_1^m, \mathscr{D}_0$ satisfy the required properties. □

By constructing $LJ_\to$-derivations $g(\mathscr{D}), (g(\mathscr{D}_i))_{i=1}^m, g(\mathscr{D}_0)$ along with $\mathscr{D}_1^m, \mathscr{D}_0$ in the above proof, we can show the following:

Theorem 25. *If Prawitz's method produces* $\mathscr{D}_1^m$, $\mathscr{D}_0$ *given input deduction* $\mathscr{D} : \Gamma, \Delta \Rightarrow C$ *and partition* $(\Gamma; \Delta)$*, then Maehara's method produces sequent derivations* $(g(\mathscr{D}_i))_{i=1}^m, g(\mathscr{D}_0)$ *given input derivation* $g(\mathscr{D})$ *and partition* $(\Gamma; \Delta)$.

We have already noted that Maehara's method provides more interpolants than Prawitz's method (section 2.3). Interpolants found by Prawitz's method are among those that satisfy a strengthening of the condition in Lemma 19.

Theorem 26. *Let* $\mathscr{D} : \Gamma, \Delta \Rightarrow C$ *be a normal deduction, and let* $(\mathscr{D}_i : \Gamma_i \Rightarrow E_i)_{i=1}^m, \mathscr{D}_0 : (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ *be the result of applying Prawitz's method to* $\mathscr{D}$ *with respect to the partition* $(\Gamma; \Delta)$*. Then every reduction sequence from*

$$\dfrac{\begin{array}{c}(z_i : E_i)_{i=1}^m, \Delta \\ \mathscr{D}_0 \\ \dfrac{C}{\overline{E_1^m \to C}} \to I, z_1^m\end{array} \qquad \begin{pmatrix}\Gamma_i \\ \mathscr{D}_i \\ E_i\end{pmatrix}_{i=1}^m}{\overline{C}} \to E$$

to $\mathscr{D}$ consists entirely of non-erasing, non-duplicating β-reduction steps.

Theorem 26 may seem similar to Theorem 8, but not all interpolants found by Maehara's method satisfy the condition in Theorem 26.

3.4 Ordering interpolants

Let us say that $\mathscr{D}_1^m$ is an $\varnothing$-*interpolant* to $\mathscr{D}\colon \Gamma \Rightarrow C$ (via $\mathscr{D}_0$) if $\mathscr{D}_1^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\varnothing)$ (via $\mathscr{D}_0$).

Lemma 27 (Substitution). *Let $(\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ be an interpolant to $\mathscr{D}\colon \Gamma,\Delta \Rightarrow C$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0\colon (z_i{:}E_i)_{i=1}^m,\Delta \Rightarrow C$. If $(\mathscr{E}_i\colon \Theta_i \Rightarrow F_i)_{i=1}^n$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{E}_0\colon (w_i : F_i)_{i=1}^n \Rightarrow E_j$, then $\mathscr{D}_1^{j-1},\mathscr{E}_1^n,\mathscr{D}_{j+1}^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\Delta)$ via $|\mathscr{D}_0[\mathscr{E}_0/z_j]|_\beta\colon (z_i{:}E_i)_{i=1}^{j-1},(w_i{:}F_i)_{i=1}^n,(z_i : E_i)_{i=j+1}^m,\Delta \Rightarrow C$.*

Proof. Conditions (I1)–(I3) of Definition 16 are clearly satisfied by $\mathscr{D}_1^{j-1},\mathscr{E}_1^n,\mathscr{D}_{j+1}^m$ and $|\mathscr{D}_0[\mathscr{E}_0/z_j]|_\beta$. As for condition (I4), Lemma 14 implies that it suffices to show that condition (I4) holds of $\mathscr{D}_0[\mathscr{E}_0/z_j]$. Consider any maximal path π in $\mathscr{D}_0[\mathscr{E}_0/z_j]$. If π starts inside an assumption $w_i : F_i$, then it must reach an occurrence inside E_j since $\mathscr{E}_0$ satisfies condition (I4). From there, π follows a path in $\mathscr{D}_0$, terminating either inside an assumption belonging to Δ or inside the endformula C, since $\mathscr{D}_0$ satisfies condition (I4). Now suppose π starts inside an assumption $z_i : E_i$ ($i \neq j$). Then π stays within $\mathscr{D}_0$ and again ends either inside an assumption belonging to Δ or inside the endformula C, for the same reason. □

Lemma 28 (Contraction). *Suppose that for some $m \geq 2$, $(\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ is an interpolant to $\mathscr{D}\colon \Gamma,\Delta \Rightarrow C$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m,\Delta \Rightarrow C$. If $\mathscr{D}_i = \mathscr{D}_j$ for some i,j such that $i \neq j$, then $\mathscr{D}_1^{j-1},\mathscr{D}_{j+1}^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0[z_i : E_i/z_j]$.*

Lemma 29 (Pruning). *Let $(\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ be an interpolant to $\mathscr{D}\colon \Gamma,\Delta \Rightarrow C$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m,\Delta \Rightarrow C$, where $m \geq 2$. If for some i,j such that $i \neq j$, $\Gamma_i = \Gamma_j$ and $\mathscr{D}_i$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{E}\colon z_i{:}E_i \Rightarrow E_j$, then $\mathscr{D}_1^{j-1},\mathscr{D}_{j+1}^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\Delta)$ via $|\mathscr{D}_0[\mathscr{E}/z_j]|_\beta$.*

Definition 30. Let $(\mathscr{E}_i\colon \Theta_i \Rightarrow E_i)_{i=1}^m$ and Let $(\mathscr{F}_i\colon \Xi_i \Rightarrow F_i)_{i=1}^n$ be two sequences of normal deductions such that $\bigcup_{i=1}^m \Theta_i = \bigcup_{i=1}^n \Xi_i$. We say that $\mathscr{E}_1^m$ is *stronger than* $\mathscr{F}_1^n$ if there are n subsets $S_1,\dots,S_n$ of $\{1,\dots,m\}$ such that

1. $S_1 \cup \cdots \cup S_n = \{1,\dots,m\}$;
2. for $j = 1,\dots,n$, $(\mathscr{E}_i)_{i\in S_j}$ is an $\varnothing$-interpolant to $\mathscr{F}_j$.

We say that $\mathscr{E}_1^m$ is *strictly stronger than* $\mathscr{F}_1^n$ if $\mathscr{E}_1^m$ is stronger than $\mathscr{F}_1^n$ and if moreover $\mathscr{F}_1^n$ is not stronger than $\mathscr{E}_1^m$.

Clearly, the relation "is stronger than" is reflexive, and Lemmas 27 and 28 imply that it is also transitive. If $\mathscr{E}_1^m$ is stronger than $\mathscr{F}_1^n$ and $\mathscr{F}_1^n$ is an interpolant to $\mathscr{D}\colon \Gamma,\Delta \Rightarrow C$ with respect to the partition $(\Gamma;\Delta)$, then $\mathscr{E}_1^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\Delta)$.

Example 31. Take the following normal deduction of (3) of Example 10:

$$\dfrac{x_1 : ((p_1 \to p_2) \to p_5) \to p_6 \qquad \dfrac{\dfrac{y_1 : p_4 \to p_5 \qquad \dfrac{x_2 : p_3 \to p_4 \qquad \dfrac{y_2 : p_2 \to p_3 \qquad \dfrac{u : p_1 \to p_2 \quad x_3 : p_1}{p_2}\,{\to}E}{p_3}\,{\to}E}{p_4}\,{\to}E}{p_5}\,{\to}E}{(p_1 \to p_2) \to p_5}\,{\to}I, u}{p_6}\,{\to}E$$

Let $\Gamma = \{x_1 : ((p_1 \to p_2) \to p_5) \to p_6,\ x_2 : p_3 \to p_4,\ x_3 : p_1\}$ and $\Delta = \{y_1 : p_4 \to p_5,\ y_2 : p_2 \to p_3\}$. The result of applying Prawitz's method to this deduction with respect to the partition $(\Gamma; \Delta)$ is $\mathscr{D}_1, \mathscr{D}_0$:

$$\mathscr{D}_1 = \dfrac{\dfrac{x_1 : ((p_1 \to p_2) \to p_5) \to p_6 \qquad \dfrac{\dfrac{v_1 : ((p_2 \to p_3) \to p_4) \to p_5 \qquad \dfrac{\dfrac{x_2 : p_3 \to p_4 \qquad \dfrac{v_2 : p_2 \to p_3 \qquad \dfrac{u : p_1 \to p_2 \quad x_3 : p_1}{p_2}\,{\to}E}{p_3}\,{\to}E}{p_4}\,{\to}E}{(p_2 \to p_3) \to p_4}\,{\to}I, v_2}{p_5}\,{\to}E}{(p_1 \to p_2) \to p_5}\,{\to}I, u}{p_6}\,{\to}E}{(((p_2 \to p_3) \to p_4) \to p_5) \to p_6}\,{\to}I, v_1$$

$$\mathscr{D}_0 = \dfrac{z_1 : (((p_2 \to p_3) \to p_4) \to p_5) \to p_6 \qquad \dfrac{\dfrac{y_1 : p_4 \to p_5 \qquad \dfrac{v : (p_2 \to p_3) \to p_4 \quad y_2 : p_2 \to p_3}{p_4}\,{\to}E}{p_5}\,{\to}E}{((p_2 \to p_3) \to p_4) \to p_5}\,{\to}I, v}{p_6}\,{\to}E$$

Another interpolant is $\mathscr{E}_1, \mathscr{E}_2$, with an auxiliary deduction $\mathscr{E}_0$:

$$\mathscr{E}_1 = x_2 : p_3 \to p_4$$

$$\mathscr{E}_2 = \dfrac{\dfrac{x_1 : ((p_1 \to p_2) \to p_5) \to p_6 \qquad \dfrac{\dfrac{v : p_2 \to p_5 \qquad \dfrac{u : p_1 \to p_2 \quad x_3 : p_1}{p_2}\,{\to}E}{p_5}\,{\to}E}{(p_1 \to p_2) \to p_5}\,{\to}I, u}{p_6}\,{\to}E}{(p_2 \to p_5) \to p_6}\,{\to}I, v$$

$$\mathscr{E}_0 = \dfrac{z_2 : (p_2 \to p_5) \to p_6 \qquad \dfrac{\dfrac{y_1 : p_4 \to p_5 \qquad \dfrac{z_1 : p_3 \to p_4 \qquad \dfrac{y_2 : p_2 \to p_3 \quad v : p_2}{p_3}\,{\to}E}{p_4}\,{\to}E}{p_5}\,{\to}E}{p_2 \to p_5}\,{\to}I, v}{p_6}\,{\to}E$$

$\mathscr{E}_1, \mathscr{E}_2$ is an $\varnothing$-interpolant to $\mathscr{D}_1$ via the following auxiliary deduction:

$$\dfrac{\dfrac{z_2 : (p_2 \to p_5) \to p_6 \qquad \dfrac{\dfrac{v_1 : ((p_2 \to p_3) \to p_4) \to p_5 \qquad \dfrac{\dfrac{z_1 : p_3 \to p_4 \qquad \dfrac{v_3 : p_2 \to p_3 \quad v_2 : p_2}{p_3}\,{\to}E}{p_4}\,{\to}E}{(p_2 \to p_3) \to p_4}\,{\to}I, v_3}{p_5}\,{\to}E}{p_2 \to p_5}\,{\to}I, v_2}{p_6}\,{\to}E}{(((p_2 \to p_3) \to p_4) \to p_5) \to p_6}\,{\to}I, v_1$$

Since $(((p_2 \to p_3) \to p_4) \to p_5) \to p_6$ does not imply either $p_3 \to p_4$ or $(p_2 \to p_5) \to p_6$, we conclude that $\mathscr{E}_1, \mathscr{E}_2$ is a strictly stronger interpolant than $\mathscr{D}_1$. Note, incidentally, that $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_0$ also satisfies the condition in Theorem 26.

Let us prove a general fact illustrated by the above example. Let $\Gamma \subseteq \mathrm{Ass}(\mathscr{D})$ and let $\mathbf{A}$ be the set of assumptions of $\mathscr{D}$ belonging to Γ. We say that Γ is *disconnected* in $\mathscr{D}$ if there is a proper subset $\mathbf{A}_1$ of $\mathbf{A}$ such that no propositional variable occurrence inside an assumption in $\mathbf{A}_1$ is linked to a propositional variable occurrence inside an assumption in $\mathbf{A} - \mathbf{A}_1$. Otherwise we say that Γ is *connected* in $\mathscr{D}$.

Lemma 32. *Let $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ be a normal deduction. Γ is disconnected in $\mathscr{D}$ if and only if there is an interpolant $\mathscr{D}_1^m$ to $\mathscr{D}$ with respect to the partition $(\Gamma; \Delta)$ with $m \geq 2$.*

Proof. The "if" direction easily follows from Lemmas 14 and 19. The "only if" direction may be proved using Lemma 15. We omit the details. $\square$

Lemma 33. *Let $\mathscr{D}_1^m$ and $\mathscr{E}_1^n$ be as in Lemma 27. If $n \geq 2$, then $\mathscr{D}_1^{j-1}, \mathscr{E}_1^n, \mathscr{D}_{j+1}^m$ is a strictly stronger interpolant than $\mathscr{D}_1^m$.*

Proof. Suppose that $\mathscr{D}_1^m$ is stronger than $\mathscr{D}_1^{j-1}, \mathscr{E}_1^n, \mathscr{D}_{j+1}^m$. Then there are subsets $S_1^{j-1}, T_1^n, S_{j+1}^m$ of $\{1, \dots, m\}$ such that $S_1 \cup \dots \cup S_{j-1} \cup T_1 \cup \dots \cup T_n \cup S_{j+1} \cup \dots \cup S_m = \{1, \dots, m\}$ and for $i = 1, \dots, j-1, j+1, \dots, m$, $(\mathscr{D}_k)_{k \in S_i}$ is an $\varnothing$-interpolant to $\mathscr{D}_i$, and for $i = 1, \dots, n$, $(\mathscr{D}_k)_{k \in T_i}$ is an $\varnothing$-interpolant to $\mathscr{E}_i$. We derive a contradiction by constructing an infinite sequence $j_0, j_1, j_2, \dots$ of elements of $\{1, \dots, m\}$ such that $j_0 = j$ and for each $k \geq 0$, $j_k \in S_{j_{k+1}}$ (which implies $\#\mathscr{D}_{j_k} \leq \#\mathscr{D}_{j_{k+1}}$ by Lemma 20), and $j_{k+1} \notin \{j_0, \dots, j_k\}$. We construct $j_0, j_1, j_2, \dots$ by induction. First set $j_0 = j$. Suppose that we have constructed $j_0, \dots, j_k$ $(k \geq 0)$. Since $\mathscr{E}_1^n$ is an $\varnothing$-interpolant to $\mathscr{D}_j$, Lemma 20 implies $\#\mathscr{E}_i < \#\mathscr{D}_j$ for each i. By the induction hypothesis, $\#\mathscr{D}_j = \#\mathscr{D}_{j_0} \leq \#\mathscr{D}_{j_k}$, which implies that $j_k \notin T_i$ for any i (again by Lemma 20). Hence there is a $j_{k+1} \in \{1, \dots, j-1, j+1, \dots, m\}$ such that $j_k \in S_{j_{k+1}}$. We have $j_{k+1} \neq j_0$, so suppose that $j_{k+1} = j_l$ for some l such that $1 \leq l \leq k$. Then $\{j_{l-1}, j_k\} \subseteq S_{j_l}$. Since $j_{l-1} \neq j_k$, this implies that $\#\mathscr{D}_{j_k} < \#\mathscr{D}_{j_l}$ by Lemma 20. But $\#\mathscr{D}_{j_l} \leq \#\mathscr{D}_{j_k}$ by induction hypothesis, a contradiction. So we have shown $j_{k+1} \notin \{j_0, \dots, j_k\}$. $\square$

Lemma 34. *If $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ is an interpolant to $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ (with respect to the partition $(\Gamma; \Delta)$) such that $\mathscr{D}_j$ is disconnected, then there is a strictly stronger interpolant to $\mathscr{D}$ than $\mathscr{D}_1^m$.*

Let us say that a deduction $\mathscr{D}$ is *connected* if $\mathrm{Ass}(\mathscr{D})$ is connected in $\mathscr{D}$.

Lemma 35. *Every normal deduction $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ has an interpolant $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ (with respect to the partition $(\Gamma; \Delta)$) such that each $\mathscr{D}_i$ is connected.*

Example 36. Let $\mathscr{D}$ be the following deduction (we omit rule labels $\to E$ and $\to I$):

$$
\dfrac{
\dfrac{y_1 : p_5 \to p_5 \to p_4 \quad \dfrac{x_1 : (p_1 \to p_4) \to p_5 \quad \dfrac{\dfrac{\dfrac{y_2 : p_2 \to p_3 \to p_4 \quad \dfrac{x_2 : p_1 \to p_2 \quad v : p_1}{p_2}}{p_3 \to p_4} \quad \dfrac{x_3 : p_1 \to p_3 \quad v : p_1}{p_3}}{p_4}}{p_1 \to p_4}\,v}{p_5}}{p_5 \to p_6}
\quad
\dfrac{x_1 : (p_1 \to p_4) \to p_5 \quad \dfrac{\dfrac{\dfrac{y_3 : p_3 \to p_2 \to p_4 \quad \dfrac{x_3 : p_1 \to p_3 \quad v : p_1}{p_3}}{p_2 \to p_4} \quad \dfrac{x_2 : p_1 \to p_2 \quad v : p_1}{p_2}}{p_4}}{p_1 \to p_4}\,v}{p_5}
}{p_6}
$$

Let $\Gamma = \{x_1 : (p_1 \to p_4) \to p_5,\ x_2 : p_1 \to p_2,\ x_3 : p_1 \to p_3\}$ and $\Delta = \{y_1 : p_5 \to p_5 \to p_6,\ y_2 : p_2 \to p_3 \to p_4,\ y_3 : p_3 \to p_2 \to p_4\}$. Given input deduction $\mathscr{D}$ and partition $(\Gamma; \Delta)$, Prawitz's method produces an interpolant of length 2:

$$
\mathscr{D}_1 = \dfrac{\dfrac{x_1 : (p_1 \to p_4) \to p_5 \quad \dfrac{\dfrac{\dfrac{u_2 : p_2 \to p_3 \to p_4 \quad \dfrac{x_2 : p_1 \to p_2 \quad v : p_1}{p_2}}{p_3 \to p_4} \quad \dfrac{x_3 : p_1 \to p_3 \quad v : p_1}{p_3}}{p_4}}{p_1 \to p_4}\, v}{p_5}}{(p_2 \to p_3 \to p_4) \to p_5}\, u_2
\qquad
\mathscr{D}_2 = \dfrac{\dfrac{x_1 : (p_1 \to p_4) \to p_5 \quad \dfrac{\dfrac{\dfrac{u_3 : p_3 \to p_2 \to p_4 \quad \dfrac{x_3 : p_1 \to p_3 \quad v : p_1}{p_3}}{p_2 \to p_4} \quad \dfrac{x_3 : p_1 \to p_2 \quad v : p_1}{p_2}}{p_4}}{p_1 \to p_4}\, v}{p_5}}{(p_3 \to p_2 \to p_4) \to p_5}\, u_3
$$

However, either of the above two deductions by itself is a strictly stronger interpolant.

The above example is an illustration of the following general fact:

Lemma 37. *Let $\mathscr{D}_1^m$ be as in Lemma 29. Then $\mathscr{D}_1^{j-1}, \mathscr{D}_{j+1}^m$ is a strictly stronger interpolant than $\mathscr{D}_1^m$.*

3.5 The new method

Definition 38. Let $\mathscr{D}_1^m$ be an interpolant to $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ with respect to the partition $(\Gamma; \Delta)$. We say that $\mathscr{D}_1^m$ is a *strongest interpolant* to $\mathscr{D}$ with respect to $(\Gamma; \Delta)$ if $\mathscr{D}_1^m$ is stronger than every interpolant to $\mathscr{D}$ with respect to $(\Gamma; \Delta)$.

It is not immediately clear whether one can always find a strongest interpolant when given a normal deduction together with a partition. We present a new method for constructing interpolants which works by induction on $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ and finds a strongest interpolant at every step. In particular, each component $\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i$ of the constructed interpolant is *connected*.

In the method we are about to describe, we make use of the following procedure, called *pruning*, which turns a sequence of deductions $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow \check{E}_i)_{i=1}^{\check{m}}, \mathscr{D}_0 \colon (\check{z}_i : \check{E}_i)_{i=1}^{\check{m}}, \Delta \Rightarrow C$ satisfying (I1)–(I4) (with respect to $\mathscr{D}$ and $(\Gamma; \Delta)$) into another such sequence $\mathscr{D}_1^m, \mathscr{D}_0$. Let

$$
\check{M} = \{\, i \mid 1 \le i \le \check{m} \text{ and there is no } j < i \text{ such that } \check{\mathscr{D}}_j \text{ is an } \varnothing\text{-interpolant to } \check{\mathscr{D}}_i \,\},
$$

and let

$$
(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m = (\check{\mathscr{D}}_i)_{i \in \check{M}}.
$$

Prepare fresh variables z_1^m of types E_1^m, respectively. For $i = 1, \dots, \check{m}$, let $\mu(i)$ be the least j such that $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_i$ (such a j always exists), and let $\mathscr{M}_i \colon z_{\mu(i)} : E_{\mu(i)} \Rightarrow \check{E}_i$ be an auxiliary deduction for $\mathscr{D}_{\mu(i)}, \check{\mathscr{D}}_i$. Then we define prune$(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$ to be $\mathscr{D}_1^m, \mathscr{D}_0$, where

$$
\mathscr{D}_0 = |\check{\mathscr{D}}_0[(\mathscr{M}_i/\check{z}_i)_{i=1}^{\check{m}}]|_\beta \colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C.
$$

This ‘definition’ does not uniquely determine $\mathscr{D}_0$ because the choice of auxiliary deduction $\mathscr{M}_i$ is not unique in general. We will later give an explicit construction of $\mathscr{M}_i$ along with an algorithm for determining whether $\check{\mathscr{D}}_j$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_i$, which is designed to work for a restricted class of deductions that are actually encountered in our method.

Lemma 39. *Let $\check{\mathscr{D}}_1^{\check{m}}$ be an interpolant to $\mathscr{D}$ with respect to $(\Gamma; \Delta)$ via $\check{\mathscr{D}}_0$, and let $\mathscr{D}_1^m, \mathscr{D}_0 = prune(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. Then*

1. *$\mathscr{D}_1^m$ is an interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0$.*

2. *$\mathscr{D}_i$ is not an $\varnothing$-interpolant to $\mathscr{D}_j$ if $1 \le i < j \le m$.*[13]

Lemma 40. *Let $(\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i)_{i=1}^m$ be an interpolant to a normal deduction $\mathscr{D}\colon \Gamma,\Delta \Rightarrow C$ with respect to the partition $(\Gamma;\Delta)$ via $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$.*

1. *$\mathscr{D}_0$ ends in $\to I$ only if $\mathscr{D}$ ends in $\to I$;*

2. *If the main branch of $\mathscr{D}$ leads to some $y : B \in \Delta$, then the main branch of $\mathscr{D}_0$ leads to $y : B$;*

3. *If the main branch of $\mathscr{D}$ leads to some $x : A \in \Gamma$, then for some i, the main branch of $\mathscr{D}_0$ leads to $z_i : E_i$ and the main branch of $\mathscr{D}_i$ leads to $x : A$.*

Definition 41. Let $\mathscr{D}_i\colon \Gamma_i \Rightarrow E_i$ be a normal deduction satisfying condition (I3) of Definition 16 and let $\mathscr{D}_0\colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ be a normal deduction satisfying condition (I4) of Definition 16 with respect to the partition $((z_i : E_i)_{i=1}^m; \Delta)$. We say that $\mathscr{D}_0$ is *long for* $\mathscr{D}_i$ (with respect to $z_i : E_i$) if $\mathscr{D}_0[\mathscr{I}/z_i] \twoheadrightarrow_\beta \mathscr{D}_0$ for every $\mathscr{I} : z_i : E_i \Rightarrow E_i$ such that $\mathscr{D}_i$ is an $\varnothing$-interpolant to itself via $\mathscr{I}$.

Example 42. Let

$$\mathscr{D}_1 = \dfrac{\dfrac{x_1 : q \to r \qquad \dfrac{u : p \to q \quad x_2 : p}{q}\,{\to}E}{r}\,{\to}E}{(p \to q) \to r}\,{\to}I,u \qquad \mathscr{D}_0 = \dfrac{y : ((p \to q) \to r) \to s \qquad \dfrac{\dfrac{z_1 : (p \to q) \to r \quad u : p \to q}{r}\,{\to}E}{(p \to q) \to r}\,{\to}I,u}{r}\,{\to}E$$

Then $\mathscr{D}_0$ is not long for $\mathscr{D}_1$ with respect to $z_1 : (p \to q) \to r$. To see this, let $\mathscr{I}$ be the η-long form of $z_1 : (p \to q) \to r$:

$$\mathscr{I} = \dfrac{\dfrac{z_1 : (p \to q) \to r \qquad \dfrac{\dfrac{u : p \to q \quad v : p}{q}\,{\to}E}{p \to q}\,{\to}I,v}{r}\,{\to}E}{(p \to q) \to r}\,{\to}I,u$$

While $\mathscr{D}_1$ is an $\varnothing$-interpolant to itself via $\mathscr{I}$, we do not have $\mathscr{D}_0[\mathscr{I}/z_1] \twoheadrightarrow_\beta \mathscr{D}_0$. The deduction

$$\widetilde{\mathscr{D}}_0 = \dfrac{y : ((p \to q) \to r) \to s \qquad \dfrac{\dfrac{z_1 : (p \to q) \to r \qquad \dfrac{u : p \to q \quad v : p}{p \to q}\,{\to}I,v}{r}\,{\to}E}{(p \to q) \to r}\,{\to}I,u}{r}\,{\to}E \qquad (= |\mathscr{D}_0[\mathscr{I}/z_1]|_\beta)$$

satisfies condition (I4) of Definition 16 with respect to $(z_1 : (p \to q) \to r; y : ((p \to q) \to r) \to s)$ and we have

$$\widetilde{\mathscr{D}}_0[\mathscr{D}_1/z_1] =_\beta \mathscr{D}_0[\mathscr{D}_1/z_1].$$

It is easy to see that $\widetilde{\mathscr{D}}_0$ is long for $\mathscr{D}_1$ with respect to $z_1 : (p \to q) \to r$.

[13] The definition of prune allows for the possibility that $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{D}_i$ for $i < j$. In our method, however, it will always be the case that $\check{\mathscr{D}}_i$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_j$ if and only if $\check{\mathscr{D}}_j$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_i$, so that $\mathscr{D}_1^m$ has no two distinct deductions such that one is an $\varnothing$-interpolant to the other.

Example 43. Let

$$\mathscr{D}_1 = \dfrac{x_1 : q \to r \qquad \dfrac{\dfrac{u : p \to p \to q \quad x_2 : p}{p \to q}\,{\to}E \qquad x_2 : p}{q}\,{\to}E}{\dfrac{r}{(p \to p \to q) \to r}\,{\to}I, u}\,{\to}E \qquad \mathscr{D}_0 = \dfrac{\dfrac{z_1 : (p \to p \to q) \to r \qquad \dfrac{\dfrac{\dfrac{\dfrac{u : p \to p \to q \quad v : p}{p \to q}\,{\to}E \quad w : p}{q}\,{\to}E}{p \to q}\,{\to}I, w}{p \to p \to q}\,{\to}I, v}{r}\,{\to}E}{(p \to p \to q) \to r}\,{\to}I, u$$

($\mathscr{D}_0$ is the η-long form of $z_1 : (p \to p \to q) \to r$.) Then $\mathscr{D}_1$ is an $\varnothing$-interpolant to itself via $\mathscr{D}_0$, but $\mathscr{D}_0$ is not long for $\mathscr{D}_1$ with respect to $z_1 : (p \to p \to q) \to r$. To see this, note that $\mathscr{D}_1$ is an $\varnothing$-interpolant to itself via

$$\mathscr{J} = \dfrac{\dfrac{z_1 : (p \to p \to q) \to r \qquad \dfrac{\dfrac{\dfrac{\dfrac{u : p \to p \to q \quad v : p}{p \to q}\,{\to}E \quad w : p}{q}\,{\to}E}{p \to q}\,{\to}I, v}{p \to p \to q}\,{\to}I, w}{r}\,{\to}E}{(p \to p \to q) \to r}\,{\to}I, u$$

but

$$\mathscr{D}_0[\mathscr{J}/z_1] \twoheadrightarrow_\beta \mathscr{J} \neq_\beta \mathscr{D}_0.$$

It is not difficult to see that there is no deduction $\mathscr{I} : z_1 : (p \to p \to q) \to r \Rightarrow (p \to p \to q) \to r$ such that $\mathscr{D}_1$ is an $\varnothing$-interpolant to itself via $\mathscr{I}$ and $\mathscr{I}$ is long for $\mathscr{D}_1$.

Lemma 44. *Suppose that $\mathscr{D}_1^m$ is an interpolant to $\mathscr{D} : \Gamma, \Delta \Rightarrow C$ with respect to $(\Gamma; \Delta)$.*

1. *There is an auxiliary deduction $\mathscr{D}_0$ for $\mathscr{D}_1^m, \mathscr{D}$ such that $\mathscr{D}$ and $\mathscr{D}_0$ have identical final blocks of applications of $\to I$; that is to say, if $\mathscr{D}$ is of the form*

$$\begin{array}{c} \Gamma, \Delta, ((u_i : A_i)^\circ)_{i=1}^n \\ \mathscr{D}^- \\ \dfrac{B}{A_1^n \to B}\,{\to}I, u_1^n \end{array}$$

   where $\mathscr{D}^-$ does not end in $\to I$, then $\mathscr{D}_0$ is of the form

$$\begin{array}{c} (z_i : E_i)_{i=1}^m, \Delta, ((u_i : A_i)^\circ)_{i=1}^n \\ \mathscr{D}_0^- \\ \dfrac{B}{A_1^n \to B}\,{\to}I, u_1^n \end{array}$$

2. *Suppose that $\mathscr{D}_0 : (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ is an auxiliary deduction for $\mathscr{D}_1^m, \mathscr{D}$ such that the main branch of $\mathscr{D}_0$ leads to $z_1 : E_1$. If $\mathscr{D}_0$ is long for $\mathscr{D}_1$ with respect to $z_1 : E_1$, then $\mathscr{D}$ and $\mathscr{D}_0$ have identical final blocks of applications of $\to I$.*

Theorem 45. *Given a normal deduction $\mathscr{D} : \Gamma, \Delta \Rightarrow C$, one can find a strongest interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma; \Delta)$.*

Proof. We first describe the construction of $\mathscr{D}_1^m, \mathscr{D}_0$ from $\mathscr{D}$, proving that $\mathscr{D}_1^m, \mathscr{D}_0$ satisfies conditions (I1)–(I4) of Definition 16.[14] We do this by induction on $\mathscr{D}$. The main difference from Prawitz's method is that in our method, assumption classes never switch sides in the partition of contexts over the course of induction and the construction of interpolants proceeds independently of the construction of auxiliary deductions. It will always be trivial to check condition (I1) of Definition 16 (see the remark following Lemma 19), so we will not bother to prove it explicitly.We construct $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m, \mathscr{D}_0 \colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ in such a way that if the main branch of $\mathscr{D}_0$ leads to some $z_i : E_i$, then $i = 1$.

Induction Basis. $\mathscr{D}$ is an assumption. This case is treated exactly as in Prawitz's method.

Induction Step.

Case 1. The last inference of $\mathscr{D}$ is $\to I$. This case is treated exactly as in Prawitz's method.

Case 2. The last inference of $\mathscr{D}$ is $\to E$. $\mathscr{D}$ is of the form

$$\dfrac{\begin{matrix}\Gamma', \Delta' \\ \mathscr{D}' \\ C'' \to C\end{matrix} \qquad \begin{matrix}\Gamma'', \Delta'' \\ \mathscr{D}'' \\ C''\end{matrix}}{C} \to E$$

where $\Gamma' \cup \Gamma'' = \Gamma$ and $\Delta' \cup \Delta'' = \Delta$. This case is broken up into *four* subcases, depending not only on where the main branch of $\mathscr{D}'$ leads to, but also on where the main branch of $\mathscr{D}''$ leads to. In each subcase, we construct $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow \check{E}_i)_{i=1}^{\check{m}}, \check{\mathscr{D}}_0 \colon (\check{z}_i : \check{E}_i)_{i=1}^{\check{m}}, \Delta \Rightarrow C$ using the induction hypothesis, and then obtain $\mathscr{D}_1^m, \mathscr{D}_0 = \mathrm{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. (The exact identity of $\mathscr{D}_0$ will be indeterminate until we completely specify the function prune.)

We first apply the induction hypothesis to $\mathscr{D}'$ with respect to the partition $(\Gamma'; \Delta')$ and obtain

$$(\mathscr{D}'_i \colon \Gamma'_i \Rightarrow F_i)_{i=1}^n, \quad \mathscr{D}'_0 \colon (w_i : F_i)_{i=1}^n, \Delta' \Rightarrow C'' \to C.$$

This will be used in all subcases. By Lemma 40, $\mathscr{D}'_0$ cannot end in $\to I$.

Case 2.1. The main branch of $\mathscr{D}'$ leads to an assumption belonging to Δ'. By Lemma 40, the main branch of $\mathscr{D}'_0$ must also lead to an assumption belonging to Δ'. Apply the induction hypothesis to $\mathscr{D}''$ with respect to the partition $(\Gamma''; \Delta'')$ and obtain

$$(\mathscr{D}''_i \colon \Gamma''_i \Rightarrow G_i)_{i=1}^p, \quad \mathscr{D}''_0 \colon (v_i : G_i)_{i=1}^p, \Delta'' \Rightarrow C''.$$

We can assume that w_1^n and v_1^p are pairwise distinct. Let

$$\check{\mathscr{D}}_1^{\check{m}} = \mathscr{D}'^{\,n}_1, \mathscr{D}''^{\,p}_1,$$

and let $\check{\mathscr{D}}_0 \colon (w_i : F_i)_{i=1}^n, (v_i : G_i)_{i=1}^p, \Delta \Rightarrow C$ be the following deduction:

$$\check{\mathscr{D}}_0 = \dfrac{\begin{matrix}(w_i : F_i)_{i=1}^n, \Delta' \\ \mathscr{D}'_0 \\ C'' \to C\end{matrix} \qquad \begin{matrix}(v_i : G_i)_{i=1}^p, \Delta'' \\ \mathscr{D}''_0 \\ C''\end{matrix}}{C} \to E$$

[14] Since we know from Maehara's and Prawitz's results that an interpolant always exists, the fact that $\mathscr{D}_1^m$ is an interpolant is a consequence of the fact that $\mathscr{D}_1^m$ is stronger than any interpolant, which we will prove later. However, it is convenient to know that $\mathscr{D}_1^m$ is an interpolant when describing the construction of $\mathscr{D}_1^m$.

Now let $\mathscr{D}_1^m, \mathscr{D}_0 = \text{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$.

We now show that $\mathscr{D}_1^m, \mathscr{D}_0$ satisfies conditions (I1)–(I4) of Definition 16. By part 1 of Lemma 39, it suffices to show the same for $\check{\mathscr{D}}^{\check{m}}, \check{\mathscr{D}}_0$. The first three conditions are easy to check. Let us check condition (I4). Note that since $\mathscr{D}_0'$ does not end in $\to I$, any maximal path in $\mathscr{D}_0'$ that starts inside the endformula $C'' \to C$ must end inside an assumption belonging to Δ'. Consider any maximal path π in $\check{\mathscr{D}}_0$ that starts inside some $w_i : F_i$. By the induction hypothesis and the property just mentioned, π must end inside an assumption belonging to Δ'. Now consider any maximal path π in $\check{\mathscr{D}}_0$ that starts inside some $v_i : G_i$. π must either stay within $\mathscr{D}_0''$ and end inside an assumption belonging to Δ'' or reach the endformula C'' of $\mathscr{D}_0''$. In the latter case, π must end in an assumption belonging to Δ' by the property mentioned above.

Case 2.2. The main branch of $\mathscr{D}'$ leads to an assumption belonging to Γ'. By Lemma 40, the main branch of $\mathscr{D}_0'$ must lead to $w_1 : F_1$. Since $\mathscr{D}_0'$ does not end in $\to I$, $\mathscr{D}_0'$ must have the following form:

$$\text{(6)} \qquad \mathscr{D}_0' = \dfrac{w_1 : C_1^k \to C'' \to C \qquad \begin{matrix} (w_i : F_i)_{i \in N}, \Delta' \\ \mathscr{C}_1^k \\ C_1^k \end{matrix}}{C'' \to C} \to E$$

where $F_1 = C_1^k \to C'' \to C$ and

$$\{1\} \cup N = \{1, \dots, n\}.$$

It is easy to see that each $\mathscr{C}_i$ satisfies the following condition, for otherwise $\mathscr{D}_0'$ would violate condition (I4):

(A) Every maximal path in $\mathscr{C}_i$ starting inside the endformula C_i or some $w_j : F_j$ must end inside an assumption belonging to Δ'.

Write $A_1^l \to B$ for C'', so that

$$\text{(7)} \qquad \mathscr{D}'' = \dfrac{\begin{matrix} \Gamma'', \Delta'', ((u_j : A_j)^\circ)_{j=1}^l \\ \mathscr{B} \\ B \end{matrix}}{A_1^l \to B} \to I, u_1^l$$

where $\mathscr{B}$ does not end in $\to I$. Apply the induction hypothesis to $\mathscr{B}$ with respect to the partition $(\Gamma'', ((u_j : A_j)^\circ)_{j=1}^l; \Delta'')$ and obtain

$$(\mathscr{B}_i \colon \Gamma_i'', ((u_j : A_j)^\circ)_{j=1}^l \Rightarrow G_i)_{i=1}^p, \quad \mathscr{B}_0 \colon (v_i : G_i)_{i=1}^p, \Delta'' \Rightarrow B,$$

where $\bigcup_{i=1}^p \Gamma_i'' = \Gamma''$, and w_1^n and v_1^p are pairwise distinct. By Lemma 40, $\mathscr{B}_0$ does not end in $\to I$.

Case 2.2.1. The main branch of $\mathscr{D}''$ leads to an assumption belonging to Δ''. By Lemma 40, the main branch of $\mathscr{B}_0$ also leads to an assumption belonging to Δ''.

Let $\hat{l}$ be the least that satisfies the following condition:

(8) For every j such that $\hat{l} + 1 \le j \le l$, there is an a_j $(1 \le a_j \le p)$ satisfying:

i. $\mathscr{B}_{a_j} = u_j : A_j$;

ii. $u_j : A_j \notin \mathrm{Ass}(\mathscr{B}_i)$ for $i \neq a_j$.

Let

$$P = \{1, \ldots, p\} - \{\, a_j \mid \hat{l} + 1 \leq j \leq l \,\},$$
$$\hat{\mathscr{B}}_0 = \mathscr{B}_0[(u_j : A_j / v_{a_j})_{j=\hat{l}+1}^{l}] \colon (v_i : G_i)_{i \in P}, (u_j : A_j)_{j=\hat{l}+1}^{l}, \Delta'' \Rightarrow B.$$

It is easy to see that $(\mathscr{B}_i)_{i \in P}$ is an interpolant to $\mathscr{B}$ with respect to the partition $(\Gamma'', ((u_j : A_j)^{\circ})_{j=1}^{\hat{l}}; (u_j : A_j)_{j=\hat{l}+1}^{l}, \Delta'')$ via $\hat{\mathscr{B}}_0$.

We have seen that $\mathscr{B}_0$ does not end in $\to I$ and the main branch of $\mathscr{B}_0$ leads to an assumption belonging to Δ''. Since $\mathscr{B}_0$ satisfies condition (I4) by induction hypothesis, we have the following:

(B) Every maximal path in $\hat{\mathscr{B}}_0$ that starts inside the endformula B, some $u_j : A_j$ ($\hat{l} + 1 \leq j \leq l$), or some $v_i : G_i$ ends inside an assumption belonging to Δ''.

Case 2.2.1.1. $\hat{l} = 0$. Let

$$\check{\mathscr{D}}_1^{\check{m}} = \mathscr{D}'^{n}_1, (\mathscr{B}_i)_{i \in P},$$

and let $\check{\mathscr{D}}_0 \colon (w_i : F_i)_{i=1}^{n}, (v_i : G_i)_{i \in P}, \Delta \Rightarrow C$ be the following deduction:

$$\check{\mathscr{D}}_0 = \dfrac{\begin{array}{c}(w_i : F_i)_{i=1}^{n}, \Delta' \\ \mathscr{D}'_0 \\ (A_1^l \to B) \to C\end{array} \qquad \dfrac{\begin{array}{c}(v_i : G_i)_{i \in P}, (u_j : A_j)_{j=1}^{l}, \Delta'' \\ \hat{\mathscr{B}}_0 \\ B\end{array}}{A_1^l \to B}\,{\to}I, u_1^l}{C}\,{\to}E \tag{9}$$

We let $\mathscr{D}_1^m, \mathscr{D}_0 = \mathrm{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$.[15]

We have to show that condition (I4) is satisfied by $\check{\mathscr{D}}_0$. This easily follows from (B).

Case 2.2.1.2. $\hat{l} \geq 1$. Let

$$P^{+} = \{\, i \in P \mid \mathrm{Ass}(\mathscr{B}_i) \text{ contains at least one of } (u_j : A_j)_{j=1}^{\hat{l}} \,\},$$
$$P^{-} = P - P^{+}.$$

Let $\hat{\mathscr{D}}_1 \colon \Gamma'_1 \cup \bigcup_{i \in P^+} \Gamma''_i \Rightarrow C_1^k \to ((G_i)_{i \in P^+} \to A_{\hat{l}+1}^{l} \to B) \to C$ be the following deduction:

(10)

$$\hat{\mathscr{D}}_1 = \dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'_1 \\ \mathscr{D}'_1 \\ C_1^k \to (A_1^l \to B) \to C\end{array} \quad (\hat{u}_j : C_j)_{j=1}^{k}}{(A_1^l \to B) \to C}\,{\to}E \qquad \dfrac{\dfrac{\hat{v} : (G_i)_{i \in P^+} \to A_{\hat{l}+1}^{l} \to B \quad \left(\begin{array}{c}\Gamma''_i, ((u_j : A_j)^{\circ})_{j=1}^{\hat{l}} \\ \mathscr{B}_i \\ G_i\end{array}\right)_{i \in P^+}}{A_{\hat{l}+1}^{l} \to B}\,{\to}E}{A_1^l \to B}\,{\to}I, u_1^{\hat{l}}}{C}\,{\to}E}{C_1^k \to ((G_i)_{i \in P^+} \to A_{\hat{l}+1}^{l} \to B) \to C}\,{\to}I, \hat{u}_1^k, \hat{v}$$

[15] The sequence $\mathscr{D}_1^m, \mathscr{D}_0$ constructed this way turns out to be the same as the result one obtains if one applies the construction of Case 2.1.

Note that $\hat{\mathscr{D}}_1$ normalizes in at most k non-erasing β-reduction steps. Let

$$\check{\mathscr{D}}_1^{\check m} = |\hat{\mathscr{D}}_1|_\beta, (\mathscr{D}'_i)_{i\in N}, (\mathscr{B}_i)_{i\in P^-}.$$

Let $\check{\mathscr{D}}_0 \colon \check z_1 : C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat l+1} \to B) \to C, (w_i : F_i)_{i\in N}, (v_i : G_i)_{i\in P^-}, \Delta \Rightarrow C$ be the following deduction:

(11)

$$\check{\mathscr{D}}_0 = \dfrac{\dfrac{\check z_1 : C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat l+1} \to B) \to C \qquad \begin{matrix}(w_i : F_i)_{i\in N}, \Delta' \\ \mathscr{C}_1^k \\ C_1^k\end{matrix}}{((G_i)_{i\in P^+} \to A^l_{\hat l+1} \to B) \to C}\to E \qquad \dfrac{\begin{matrix}(v_i : G_i)_{i\in P^+}, (v_i : G_i)_{i\in P^-}, (u_j : A_j)^l_{j=\hat l+1}, \Delta'' \\ \hat{\mathscr{B}}_0 \\ B\end{matrix}}{(G_i)_{i\in P^+} \to A^l_{\hat l+1} \to B}\to I, (v_i)_{i\in P^+}, u^l_{\hat l+1}}{C}\to E$$

where $\mathscr{C}_1^k$ is as in (6). Now let $\mathscr{D}_1^m, \mathscr{D}_0 = \operatorname{prune}(\check{\mathscr{D}}_1^{\check m}, \check{\mathscr{D}}_0)$.[16]

Let us show that $\check{\mathscr{D}}_1^{\check m}, \check{\mathscr{D}}_0$ satisfies conditions (I1)–(I4) of Definition 16. Since $\hat{\mathscr{D}}_1$ reduces to $\check{\mathscr{D}}_1$ by non-erasing β-reduction steps, Lemma 14 implies that it suffices to show that these conditions are satisfied by $\hat{\mathscr{D}}_1, \check{\mathscr{D}}_2^m, \check{\mathscr{D}}_0$. Condition (I1) is obvious. That condition (I2) is satisfied can be seen as follows:

$$\check{\mathscr{D}}_0[\hat{\mathscr{D}}_1/\check z_1, (\mathscr{D}'_i/w_i)_{i\in N}, (\mathscr{B}_i/v_i)_{i\in P^-}]$$

$\twoheadrightarrow_\beta$

$$\dfrac{\dfrac{\begin{matrix}\Gamma'_1 \\ \mathscr{D}'_1 \\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \qquad \begin{matrix}\bigcup_{i\in N} \Gamma'_i, \Delta' \\ \mathscr{C}_1^k[(\mathscr{D}'_i/w_i)_{i\in N}] \\ C_1^k\end{matrix}}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}(v_i : G_i)_{i\in P^+}, \bigcup_{i\in P^-} \Gamma''_i, (u_j : A_j)^l_{j=\hat l+1}, \Delta'' \\ \hat{\mathscr{B}}_0[(\mathscr{B}_i/v_i)_{i\in P^-}] \\ B\end{matrix}}{(G_i)_{i\in P^+} \to A^l_{\hat l+1} \to B}\to I, (v_i)_{i\in P^+}, u^l_{\hat l+1} \qquad \left(\begin{matrix}\Gamma''_i, ((u_j : A_j)^\circ)^{\hat l}_{j=1} \\ \mathscr{B}_i \\ G_i\end{matrix}\right)_{i\in P^+}}{A^l_{\hat l+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\hat l}}{C}\to E$$

$\twoheadrightarrow_\beta$

$$\dfrac{\dfrac{\begin{matrix}\Gamma'_1 \\ \mathscr{D}'_1 \\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \qquad \begin{matrix}\bigcup_{i\in N} \Gamma'_i, \Delta' \\ \mathscr{C}_1^k[(\mathscr{D}'_i/w_i)_{i\in N}] \\ C_1^k\end{matrix}}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\begin{matrix}\bigcup_{i\in P^+} \Gamma''_i, ((u_j : A_j)^\circ)^{\hat l}_{j=1}, \bigcup_{i\in P^-} \Gamma''_i, (u_j : A_j)^l_{\hat l+1}, \Delta'' \\ \hat{\mathscr{B}}_0[(\mathscr{B}_i/v_i)_{i\in P^+}, (\mathscr{B}_i/v_i)_{i\in P^-}] \\ B\end{matrix}}{A^l_{\hat l+1} \to B}\to I, u^l_{\hat l+1}}{A_1^l \to B}\to I, u_1^{\hat l}}{C}\to E$$

=

$$\dfrac{\begin{matrix}\Gamma', \Delta' \\ \mathscr{D}_0[(\mathscr{D}'_i/w_i)^n_{i=1}] \\ (A_1^l \to B) \to C\end{matrix} \qquad \dfrac{\begin{matrix}\Gamma'', ((u_j : A_j)^\circ)^l_{j=1}, \Delta'' \\ \hat{\mathscr{B}}_0[(\mathscr{B}_i/v_i)^p_{i=1}] \\ B\end{matrix}}{A_1^l \to B}\to I, u_1^l}{C}\to E$$

$\twoheadrightarrow_\beta$ (by induction hypothesis)

$$\dfrac{\begin{matrix}\Gamma', \Delta' \\ \mathscr{D}' \\ (A_1^l \to B) \to C\end{matrix} \qquad \dfrac{\begin{matrix}\Gamma'', ((u_j : A_j)^\circ)^l_{j=1}, \Delta'' \\ \mathscr{B} \\ B\end{matrix}}{A_1^l \to B}\to I, u_1^l}{C}\to E \quad = \mathscr{D}.$$

[16] Two remarks about this construction. One can apply this construction to Case 2.2.1.1, producing a weaker interpolant. If one uses $\widetilde{P}^+$ such that $P^+ \subset \widetilde{P}^+ \subseteq P$ in place of P^+ in this construction, one still gets an interpolant, but then $\mathscr{D}_1$ becomes disconnected.

As for condition (I3), induction hypothesis takes care of $(\mathscr{D}'_i)_{i\in N}, (\mathscr{B}_i)_{i\in P^-}$, so it remains to check $\hat{\mathscr{D}}_1$. Let π be a maximal path in $\hat{\mathscr{D}}_1$ that starts inside its endformula. If π starts inside C_1^k, it passes through the endformula of $\mathscr{D}'_1$ and ends inside an assumption belonging to Γ'_1. If π starts inside $(G_i)_{i\in P^+}$, it enters some $\mathscr{B}_i$ ($i \in P^+$) through its endformula and either ends inside an assumption belonging to Γ''_i or exits $\mathscr{B}_i$ through some $u_j : A_j$ ($1 \leq j \leq \hat{l}$). In the latter case, π then travels a link associated with the last $\to E$ step, enters $\mathscr{D}'_1$ through its endformula, and ends inside an assumption belonging to Γ'_1. If π starts inside $A^l_{\hat{l}+1} \to B$, it travels a link associated with the last $\to E$ step and enters $\mathscr{D}'_1$ through its endformula, ending inside an assumption belonging to Γ'_1. If π starts inside C, it directly enters $\mathscr{D}'_1$ through its endformula and ends inside an assumption belonging to Γ'_1.

To show that condition (I4) is satisfied, consider any maximal path π in $\check{\mathscr{D}}_0$ that starts inside an assumption belonging to $\check{z}_1 : C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B) \to C, (w_i : F_i)_{i\in N}, (v_i : G_i)_{i\in P^-}$. If π starts inside an assumption belonging to $(w_i : F_i)_{i\in N}$, then, by (A), π stays within some $\mathscr{C}_i$ and ends inside an assumption belonging to Δ'. If π starts inside an assumption belonging to $(v_i : G_i)_{i\in P^-}$, then, by (B), π stays within $\hat{\mathscr{B}}_0$ and ends inside an assumption belonging to Δ''. Now suppose that π starts inside $\check{z}_1 : C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B) \to C$. If π starts inside C_1^k, then it enters some $\mathscr{C}_i$ through its endformula and ends inside an assumption belonging to Δ', by (A). If π starts inside $(G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B$, then it enters $\hat{\mathscr{B}}_0$ through some $v_i : G_i$ ($i \in P^+$), some $u_j : A_j$ ($\hat{l} + 1 \leq j \leq l$), or its endformula B, and in all three cases ends inside an assumption belonging to Δ'', by (B). If π starts inside C, then it ends inside the endformula C of $\check{\mathscr{D}}_0$.

Case 2.2.2. The main branch of $\mathscr{D}''$ leads to an assumption belonging to Γ'' or to some $u_j : A_j$. By Lemma 40, the main branch of $\mathscr{B}_0$ must lead to $v_1 : G_1$. Since $\mathscr{B}_0$ does not end in $\to I$, G_1 must have the form $H_1^q \to B$, and $\mathscr{B}_0$ must have the following form:

$$\text{(12)} \qquad \mathscr{B}_0 = \cfrac{v_1 : H_1^q \to B \qquad \begin{pmatrix} (v_j : G_j)_{j\in P_i}, \Delta''_i \\ \mathscr{H}_i \\ H_i \end{pmatrix}_{i=1}^{q}}{B}\,\to E$$

where

$$\{1\} \cup P_1 \cup \cdots \cup P_q = \{1, \ldots, p\},$$
$$\Delta''_1 \cup \cdots \cup \Delta''_q = \Delta''.$$

Since $\mathscr{B}_0$ satisfies condition (I4) of Definition 16, each $\mathscr{H}_i$ satisfies the following condition:

(C) Every maximal path in $\mathscr{H}_i$ that starts inside the endformula H_i or some $v_j : G_j$ must end inside an assumption belonging to Δ''_i.

For $i = 1, \ldots, q$, let

$$P_i^+ = \{\, j \in P_i \mid \mathrm{Ass}(\mathscr{B}_j) \text{ contains a least one of } (u_j : A_j)_{j=1}^l \,\},$$
$$P_i^- = P_i - P_i^+,$$

and then let

$$P^+ = P_1^+ \cup \cdots \cup P_q^+,$$
$$P^- = P_1^- \cup \cdots \cup P_q^-.$$

Let $\hat{\mathscr{D}}_1 \colon \Gamma_1' \cup \Gamma_1'' \cup \bigcup_{i\in P^+} \Gamma_i'' \Rightarrow C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C$ be the following deduction:

(13)

$$\hat{\mathscr{D}}_1 = \dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma_1'\\ \mathscr{D}_1'\\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \quad (\hat{u}_j : C_j)_{j=1}^k}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\begin{matrix}\Gamma_1'', ((u_j : A_j)^\circ)_{j=1}^l\\ \mathscr{B}_1\\ H_1^q \to B\end{matrix} \quad \left(\dfrac{\hat{v}_i : (G_j)_{j\in P_i^+} \to H_i \quad \left(\begin{matrix}\Gamma_j'', ((u_h : A_h)^\circ)_{h=1}^l\\ \mathscr{B}_j\\ G_j\end{matrix}\right)_{j\in P_i^+}}{H_i}\to E\right)_{i=1}^q}{B}\to E}{A_1^l \to B}\to I, u_1^l}{C}\to E}{C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C}\to I, \hat{u}_1^k, \hat{v}_1^q$$

Note that $\hat{\mathscr{D}}_1$ normalizes in at most $k+q$ non-erasing β-reduction steps. Let

$$\check{\mathscr{D}}_1^{\check{m}} = |\hat{\mathscr{D}}_1|_\beta, (\mathscr{D}_i')_{i\in N}, (\mathscr{B}_i)_{i\in P^-}.$$

Let $\check{\mathscr{D}}_0 \colon \check{z}_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C, (w_i : F_i)_{i\in N}, (v_i : G_i)_{i\in P^-}, \Delta \Rightarrow C$ be the following deduction:

(14)

$$\check{\mathscr{D}}_0 = \dfrac{\dfrac{\check{z}_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C \quad \begin{matrix}(w_i : F_i)_{i\in N}, \Delta'\\ \mathscr{C}_1^k\\ C_1^k\end{matrix}}{((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C}\to E \qquad \left(\dfrac{\begin{matrix}(v_j : G_j)_{j\in P_i^+}, (v_j : G_j)_{j\in P_i^-}, \Delta_i''\\ \mathscr{H}_i\\ H_i\end{matrix}}{(G_j)_{j\in P_i^+} \to H_i}\to I, (v_j)_{j\in P_i^+}\right)_{i=1}^q}{C}\to E$$

where $\mathscr{C}_1^k$ and $\mathscr{H}_i$ are as in (6) and (12), respectively. Now let $\mathscr{D}_1^m, \mathscr{D}_0 = \mathrm{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$.

We show that $\hat{\mathscr{D}}_1, (\mathscr{D}_i')_{i\in N}, (\mathscr{B}_i)_{i\in P^-}, \check{\mathscr{D}}_0$ satisfies conditions (I2)–(I4) of Definition 16. We start with condition (I2):

$$\check{\mathscr{D}}_0[\hat{\mathscr{D}}_1/\check{z}_1, (\mathscr{D}_i'/w_i)_{i\in N}, (\mathscr{B}_i/v_i)_{i\in P^-}]$$

$\twoheadrightarrow_\beta$

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1'\\ \mathscr{D}_1'\\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \quad \begin{matrix}\bigcup_{i\in N} \Gamma_i', \Delta'\\ \mathscr{C}_1^k[(\mathscr{D}_i'/w_i)_{i\in N}]\\ C_1^k\end{matrix}}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\begin{matrix}\Gamma_1'', ((u_j : A_j)^\circ)_{j=1}^l\\ \mathscr{B}_1\\ H_1^q \to B\end{matrix} \quad \left(\dfrac{\dfrac{\begin{matrix}(v_j : G_j)_{j\in P_i^+}, \bigcup_{j\in P_i^-} \Gamma_j'', \Delta_i''\\ \mathscr{H}_i[(\mathscr{B}_j/v_j)_{j\in P_i^-}]\\ H_i\end{matrix}}{(G_j)_{j\in P_i^+} \to H_i}\to I, (v_j)_{j\in P_i^+} \quad \left(\begin{matrix}\Gamma_j'', ((u_h : A_h)^\circ)_{h=1}^l\\ \mathscr{B}_j\\ G_j\end{matrix}\right)_{j\in P_i^+}}{H_i}\to E\right)_{i=1}^q}{B}\to E}{A_1^l \to B}\to I, u_1^l}{C}\to E$$

$\twoheadrightarrow_\beta$

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1'\\ \mathscr{D}_1'\\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \quad \begin{matrix}\bigcup_{i\in N} \Gamma_i', \Delta'\\ \mathscr{C}_1^k[(\mathscr{D}_i'/w_i)_{i\in N}]\\ C_1^k\end{matrix}}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\begin{matrix}\Gamma_1'', ((u_j : A_j)^\circ)_{j=1}^l\\ \mathscr{B}_1\\ H_1^q \to B\end{matrix} \quad \left(\begin{matrix}\bigcup_{j\in P_i} \Gamma_j'', ((u_h : A_h)^\circ)_{h=1}^l, \Delta_i''\\ \mathscr{H}_i[(\mathscr{B}_j/v_j)_{j\in P_i}]\\ H_i\end{matrix}\right)_{i=1}^q}{B}\to E}{A_1^l \to B}\to I, u_1^l}{C}\to E$$

=

$$\dfrac{\begin{array}{c}\bigcup_{i=1}^{n}\Gamma'_i,\Delta'\\ \mathscr{D}'_0[(\mathscr{D}'_i/w_i)_{i=1}^{n}]\\ (A_1^l\to B)\to C\end{array}\qquad\begin{array}{c}\bigcup_{i=1}^{p}\Gamma''_i,((u_j:A_j)^\circ)_{j=1}^{l},\Delta''\\ \mathscr{B}_0[(\mathscr{B}_i/v_i)_{i=1}^{p}]\\ \dfrac{B}{A_1^l\to B}\,{\to}I,u_1^l\end{array}}{C}\,{\to}E$$

$\twoheadrightarrow_\beta$ (by induction hypothesis)

$$\dfrac{\begin{array}{c}\Gamma',\Delta'\\ \mathscr{D}'\\ (A_1^l\to B)\to C\end{array}\qquad\begin{array}{c}\Gamma'',\Delta'',((u_j:A_j)^\circ)_{j=1}^{l}\\ \mathscr{B}\\ \dfrac{B}{A_1^l\to B}\,{\to}I,u_1^l\end{array}}{C}\,{\to}E\qquad=\mathscr{D}$$

Next we need to show that condition (I3) of Definition 16 holds of $\hat{\mathscr{D}}_1$. Let π be a maximal path in $\hat{\mathscr{D}}_1$ that starts inside the endformula $C_1^k\to((G_j)_{j\in P_i^+}\to H_i)_{i=1}^q\to C$. If π starts inside C_1^k, then π enters $\mathscr{D}'_1$ through its endformula and ends inside an assumption belonging to Γ'_1. If π starts inside some $(G_j)_{j\in P_i^+}$, then π passes through $\hat{v}_i:(G_j)_{j\in P_i^+}\to H_i$, enters some $\mathscr{B}_j$ ($j\in P_i^+$) through its endformula, and either ends inside an assumption belonging to Γ''_j or exits $\mathscr{B}_j$ through some $u_j:A_j$. If the latter, π travels a link associated with the last $\to E$ step and enters $\mathscr{D}'_1$ through its endformula, ending inside an assumption belonging to Γ'_1. If π starts inside some H_i, then π passes through $\hat{v}_i:(G_j)_{j\in P_i^+}\to H_i$, enters $\mathscr{B}_1$ through its endformula, and either ends inside an assumption belonging to Γ''_1 or exits $\mathscr{B}_1$ through some $u_j:A_j$. If the latter, π travels a link associated with the last $\to E$ step and enters $\mathscr{D}'_1$ through its endformula, ending inside an assumption belonging to Γ'_1. If π starts inside C, then it directly enters $\mathscr{D}'_1$ through its endformula and ends inside an assumption belonging to Γ'_1.

Finally, we show that $\check{\mathscr{D}}_0$ satisfies condition (I4) of Definition 16. Let π be a maximal path in $\check{\mathscr{D}}_0$ that starts inside an assumption belonging to $\check{z}_1:C_1^k\to((G_j)_{j\in P_i^+}\to H_i)_{i=1}^q\to C,(w_i:F_i)_{i\in N},(v_i:G_i)_{i\in P^-}$. If π starts inside an assumption belonging to $(w_i:F_i)_{i\in N}$, then, by (A), π stays within some $\mathscr{C}_i$ and ends inside an assumption belonging to Δ'. If π starts inside an assumption belonging to $(v_i:G_i)_{i\in P^-}$, then, by (C), π stays within some $\mathscr{H}_i$ and ends inside an assumption belonging to Δ''. Now suppose that π starts inside $\check{z}_1:C_1^k\to((G_j)_{j\in P_i^+}\to H_i)_{i=1}^q\to C$. If π starts inside C_1^k, then π enters some $\mathscr{C}_i$ through its endformula and ends inside an assumption belonging to Δ', by (A). If π starts inside some $(G_j)_{j\in P_i^+}\to H_i$, then π enters $\mathscr{H}_i$ through some $v_j:G_j$ or its endformula H_i, and in both cases ends inside an assumption belonging to Δ''_i, by (C). If π starts inside C, it ends inside the endformula C of $\check{\mathscr{D}}_0$.

This completes the description of the new method and the proof that it always outputs an interpolant together with an auxiliary deduction for it. We next prove some facts about deductions that can be components of interpolants constructed by the new method. Let

$$\mathbb{D}=\bigcup\{\{\mathscr{D}_1,\ldots,\mathscr{D}_m\}\mid\mathscr{D}_1^m,\mathscr{D}_0\text{ is a possible output of the new method}\}$$

Note that all deductions $\check{\mathscr{D}}_i$ that are constructed in Case 2 of the Induction Step of the new method are in $\mathbb{D}$.

Claim A. Let $\check{\mathscr{D}}$ be a deduction in $\mathbb{D}$.

1. $\check{\mathscr{D}}$ is connected.

2. Suppose that $\check{\mathscr{D}}$ is of the form

$$\dfrac{\begin{array}{c}\check{\Gamma}, (\hat{u}_i : C_i)_{i=1}^k \\ \check{\mathscr{D}}^- \\ C\end{array}}{C_1^k \to C} \to I, \hat{u}_1^k$$

where $\check{\mathscr{D}}^-$ does not end in $\to I$. Then

   - For each $i = 1, \ldots, k$, there is exactly one assumption of the form $\hat{u}_i : C_i$ in $\check{\mathscr{D}}^-$.
   - If the maximal subdeduction of $\check{\mathscr{D}}^-$ which does not end in $\to I$ and whose main branch leads to $\hat{u}_i : C_i$ is

$$\dfrac{\hat{u}_i : (C_{i,j})_{j=1}^{r_i} \to C_{i,0} \qquad \begin{array}{c}\Sigma_{i,j} \\ \mathscr{C}_{i,j} \\ C_{i,j}\end{array}}{C_{i,0}} \to E$$

     where $C_i = (C_{i,j})_{j=1}^{r_i} \to C_{i,0}$, then
     - each $\mathscr{C}_{i,j}$ is in $\mathbb{D}$;
     - $\Sigma_{i,j}$ does not contain any $\hat{u}_h : C_h$ but contains some assumption discharged in $\check{\mathscr{D}}^-$; and
     - $\mathscr{C}_{i,j}$ is not an $\varnothing$-interpolant to $\mathscr{C}_{i,h}$ if $j < h$.

3. If every maximal path in $\check{\mathscr{D}}$ that starts inside an assumption ends inside the endformula $\check{E}$, then $\check{\mathscr{D}}$ is an assumption.

All three properties can be easily checked by induction.

Claim B. Let $\check{\mathscr{D}}$ be a deduction in $\mathbb{D}$. Let a normal deduction $\widetilde{\mathscr{D}} : \check{\Gamma} \Rightarrow \widetilde{E}$ be given. Then

1. One can determine whether $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$, and if so, produce a deduction $\mathscr{M} : \widetilde{z} : \widetilde{E} \Rightarrow \check{E}$ such that

   (a) $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ via $\mathscr{M}$; and

   (b) $\mathscr{M}$ is long for $\widetilde{\mathscr{D}}$.

2. If $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$, then $\check{\mathscr{D}}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}$.

We prove the claim by induction on the construction of $\check{\mathscr{D}}$.

Induction Basis. $\check{\mathscr{D}}$ is first constructed in Case 1 of the Induction Basis of the new method, i.e., $\check{\mathscr{D}} = x : C$. Then the only interpolant to $\check{\mathscr{D}}$ is $\check{\mathscr{D}}$ itself, and the only auxiliary deduction for $\check{\mathscr{D}}, \check{\mathscr{D}}$, up to the choice of variable, is $\widetilde{z} : C$. If $\widetilde{\mathscr{D}} = \check{\mathscr{D}}$, we let $\mathscr{E} = \widetilde{z} : C$. Clearly $\mathscr{E}[\mathscr{E}/\widetilde{z}] = \mathscr{E}$ and all the conditions are satisfied.

Induction Step.

Case 1. $\check{\mathscr{D}}$ is first constructed in Case 2.2.1.2 of the Induction Step of the new method, i.e., $\check{\mathscr{D}}$ is the normal form of (10), repeated below:

$$(10)\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma'_1\\ \mathscr{D}'_1\\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \quad (\hat{u}_j : C_j)_{j=1}^k}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\hat{v} : (G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B \quad \begin{pmatrix}\Gamma''_i, ((u_j : A_j)^\circ)_{j=1}^{\hat{l}}\\ \mathscr{B}_i\\ G_i\end{pmatrix}_{i\in P^+}}{A^l_{\hat{l}+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\hat{l}}}{C}\to E}{C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B) \to C}\to I, \hat{u}_1^k, \hat{v}$$

Let $\hat{k}$ ($0 \le \hat{k} \le k$) be such that

$$(15)\qquad \mathscr{D}'_1 = \dfrac{\begin{matrix}\Gamma'_1, (\hat{u}_i : C_i)_{i=1}^{\hat{k}}\\ \mathscr{D}'^{-}_1\\ C^k_{\hat{k}+1} \to (A_1^l \to B) \to C\end{matrix}}{C_1^k \to (A_1^l \to B) \to C}\to I, \hat{u}_1^{\hat{k}}$$

where $\mathscr{D}'^{-}_1$ does not end in $\to I$. Then

$$\check{\mathscr{D}} = \dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma'_1, (\hat{u}_i : C_i)_{i=1}^{\hat{k}}\\ \mathscr{D}'^{-}_1\\ C^k_{\hat{k}+1} \to (A_1^l \to B) \to C\end{matrix} \quad (\hat{u}_i : C_i)_{i=\hat{k}+1}^k}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\hat{v} : (G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B \quad \begin{pmatrix}\Gamma''_i, ((u_j : A_j)^\circ)_{j=1}^{\hat{l}}\\ \mathscr{B}_i\\ G_i\end{pmatrix}_{i\in P^+}}{A^l_{\hat{l}+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\hat{l}}}{C}\to E}{C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B) \to C}\to I, \hat{u}_1^k, \hat{v}$$

We can show that $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ if and only if $\widetilde{\mathscr{D}}$ is the normal form of a deduction of the form

(16)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma'_1\\ \widetilde{\mathscr{D}}'_1\\ \widetilde{C}_1^k \to (A_1^l \to B) \to C\end{matrix} \quad (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^k}{(A_1^l \to B) \to C}\to E \qquad \dfrac{\dfrac{\widetilde{u}_{k+1} : \widetilde{G}_1^{|P^+|} \to A^l_{\hat{l}+1} \to B \quad \begin{pmatrix}\Gamma''_{\rho(i)}, ((u_j : A_j)^\circ)_{j=1}^{\hat{l}}\\ \widetilde{\mathscr{G}}_i\\ \widetilde{G}_i\end{pmatrix}_{i=1}^{|P^+|}}{A^l_{\hat{l}+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\hat{l}}}{C}\to E}{(\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C}\to I, (\widetilde{u}_{\pi(i)})_{i=1}^{k+1}$$

where π is a permutation of $\{1, \ldots, k+1\}$, $\widetilde{\mathscr{D}}'_1$ is an $\varnothing$-interpolant to $\mathscr{D}'_1$, $\widetilde{C}_{k+1} = \widetilde{G}_1^{|P^+|} \to A^l_{\hat{l}+1} \to B$, ρ is a bijection from $\{1, \ldots, |P^+|\}$ to P^+, and for $i = 1, \ldots, |P^+|$, $\widetilde{\mathscr{G}}_i$ is an $\varnothing$-interpolant to $\mathscr{B}_{\rho(i)}$.

We first prove the "only if" direction of this statement. Suppose that $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ via $\mathscr{E}$. Since $\check{\mathscr{D}}$ is connected by part 1 of Claim A, Lemma 32 implies that $\mathscr{E}$ can have only one assumption. By part 1 of Lemma 44, we may assume that $\mathscr{E}$ is of the following form:

$$\mathscr{E} = \dfrac{\dfrac{\widetilde{z} : \widetilde{F}_1^{\widetilde{k}} \to C \quad \begin{pmatrix}\Theta_i\\ \widetilde{\mathscr{F}}_i\\ \widetilde{F}_i\end{pmatrix}_{i=1}^{\widetilde{k}}}{C}\to E}{C_1^k \to ((G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B) \to C}\to I, \hat{u}_1^k, \hat{v}$$

where

$$\Theta_1 \cup \cdots \cup \Theta_{\widetilde{k}} = (\hat{u}_i : C_i)_{i=1}^{k}, \hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B.$$

Since $\mathscr{E}$ satisfies condition (I4) of Definition 16 (with respect to $(\widetilde{z} : \widetilde{F}_1^{\widetilde{k}} \to C; \varnothing)$), each $\widetilde{\mathscr{F}_i}$ must satisfy condition (I3) of Definition 16. This implies that $\Theta_i \neq \varnothing$ for each i. Moreover, by part 2 of Claim A and Lemma 19, it is not difficult to see that

$$|\Theta_i| = 1 \quad \text{for each } i,$$
$$\Theta_i \cap \Theta_j = \varnothing \quad \text{if } i \neq j.$$

So $\widetilde{k} = k+1$. By part 3 of Lemma 40, the main branch of $\widetilde{\mathscr{D}}$ leads to an assumption belonging to Γ'_1. Now we show that $\widetilde{\mathscr{D}}$ must end in at least $\widetilde{k}$ applications of $\to I$. Suppose not. Then, since $\mathscr{E}[\widetilde{\mathscr{D}}/\widetilde{z}] \twoheadrightarrow_\beta \check{\mathscr{D}}$, the subdeduction of (10) whose endformula is $A^l_1 \to B$ must be $\widetilde{\mathscr{F}_{\widetilde{k}}}$. Then, for $i \in P^+$, $\mathrm{Ass}(\mathscr{B}_i) \subseteq \{u_j : A_j \mid 1 \leq j \leq \hat{l}\}$. Since $\widetilde{\mathscr{F}_{\widetilde{k}}}$ satisfies (I3) and each $\mathscr{B}_i$ is connected by part 1 of Claim A, each $\mathscr{B}_i$ has only one assumption and every maximal path in $\mathscr{B}_i$ that starts inside its only assumption must end inside its endformula. By part 3 of Claim A, it follows that $\mathscr{B}_i = u_j : A_j$ for some j such that $1 \leq j \leq \hat{l}$. Since $\mathscr{B}^p_1, \mathscr{B}_0$ is an output of the pruning procedure, $\mathscr{B}_i \neq \mathscr{B}_j$ for $i \neq j$, by part 2 of Lemma 39. Hence for each $j = 1, \ldots, \hat{l}$, there is a unique i such that $\mathscr{B}_i = u_j : A_j$. By the definition of $\hat{l}$, this contradicts the assumption that $\hat{l} \geq 1$. Therefore, $\widetilde{\mathscr{D}}$ must end in $\widetilde{k}$ applications of $\to I$. This means that $\widetilde{\mathscr{D}}$ must be of the following form:

$$\widetilde{\mathscr{D}} = \cfrac{\cfrac{\begin{array}{c}\Gamma'_1, (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^k \\ \widetilde{\mathscr{D}}' \\ (A^l_1 \to B) \to C\end{array} \qquad \begin{array}{c}\bigcup_{i \in P^+} \Gamma''_i, \widetilde{u}_{k+1} : \widetilde{C}_{k+1} \\ \widetilde{\mathscr{D}}'' \\ A^l_1 \to B\end{array}}{C}\,\to E}{(\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C}\,\to I, (\widetilde{u}_{\pi(i)})_{i=1}^{k+1}$$

where π is a permutation of $\{1, \ldots, k+1\}$, $\widetilde{F}_i = \widetilde{C}_{\pi(i)}$, and

$$\Theta_i = \begin{cases} \hat{u}_{\pi(i)} : C_{\pi(i)} & \text{if } 1 \leq \pi(i) \leq k, \\ \hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B & \text{if } \pi(i) = k+1. \end{cases}$$

We have to show that $\widetilde{\mathscr{D}}'$ and $\widetilde{\mathscr{D}}''$ have the required form. Let $\widetilde{\mathscr{C}_i} = \widetilde{\mathscr{F}}_{\pi^{-1}(i)}$, so that

$$\widetilde{\mathscr{C}_i} : \hat{u}_i : C_i \Rightarrow \widetilde{C}_i \quad \text{for } i = 1, \ldots, k,$$
$$\widetilde{\mathscr{C}}_{k+1} : \hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B \Rightarrow \widetilde{C}_{k+1}.$$

Let us first consider $\widetilde{\mathscr{D}}'$. Since $\mathscr{E}[\widetilde{\mathscr{D}}/\widetilde{z}] \twoheadrightarrow_\beta \check{\mathscr{D}}$,

$$\widetilde{\mathscr{D}}'[(\widetilde{\mathscr{C}_i}/\widetilde{u}_i)_{i=1}^k] \twoheadrightarrow_\beta \cfrac{\begin{array}{c}\Gamma'_1, (\hat{u}_i : C_i)_{i=1}^{\hat{k}} \\ \mathscr{D}_1'^{-} \\ C^k_{\hat{k}+1} \to (A^l_1 \to B) \to C\end{array} \quad (\hat{u}_i : C_i)_{i=\hat{k}+1}^k}{(A^l_1 \to B) \to C}\,\to E$$

This means that, for $i = \hat{k}+1, \ldots, k$, $\hat{u}_i : C_i$ cannot appear as the major premise of $\to E$ in $\widetilde{C}_i$. Therefore, for $i = \hat{k}+1, \ldots, k$,

$$\widetilde{C}_i = C_i, \qquad \widetilde{\mathscr{C}_i} = \hat{u}_i : C_i,$$

and $\widetilde{\mathscr{D}}'$ must have the following form:

$$\widetilde{\mathscr{D}}' = \begin{array}{c} \begin{array}{c} \Gamma_1', (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^{\hat{k}} \\ \widetilde{\mathscr{D}}_1'^{-} \\ C_{\hat{k}+1}^k \to (A_1^l \to B) \to C \end{array} \quad (\widetilde{u}_i : \widetilde{C}_i)_{i=\hat{k}+1}^k \\ \hline\hline (A_1^l \to B) \to C \end{array} \to E$$

It follows that

$$\widetilde{\mathscr{D}}'[(\widetilde{\mathscr{C}}_i/\widetilde{u}_i)_{i=1}^{\hat{k}}] \twoheadrightarrow_\beta \mathscr{D}_1'^{-}.$$

Let

$$\widetilde{\mathscr{D}}_1' = \begin{array}{c} \Gamma_1', (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^{\hat{k}} \\ \widetilde{\mathscr{D}}_1'^{-} \\ C_{\hat{k}+1}^k \to (A_1^l \to B) \to C \\ \hline\hline \widetilde{C}_1^k \to (A_1^l \to B) \to C \end{array} \to I, \widetilde{u}_1^{\hat{k}}$$

Then

$$\begin{array}{c} \begin{array}{c} \Gamma_1' \\ \widetilde{\mathscr{D}}_1' \\ \widetilde{C}_1^k \to (A_1^l \to B) \to C \end{array} \quad (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^k \\ \hline\hline (A_1^l \to B) \to C \end{array} \to E \quad \twoheadrightarrow_\beta \quad \widetilde{\mathscr{D}}'$$

and it is easy to see that $\widetilde{\mathscr{D}}_1'$ is an interpolant to $\mathscr{D}_1'$ via

$$\mathscr{E}' = \begin{array}{c} \widetilde{w} : \widetilde{C}_1^k \to (A_1^l \to B) \to C \quad \begin{pmatrix} \hat{u}_i : C_i \\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i \end{pmatrix}_{i=1}^{\hat{k}} \\ \hline\hline \begin{array}{c} C_{\hat{k}+1}^k \to (A_1^l \to B) \to C \\ \hline\hline C_1^k \to (A_1^l \to B) \to C \end{array} \to I, \hat{u}_1^{\hat{k}} \end{array} \to E$$

Let us now turn to $\widetilde{\mathscr{D}}''$. Since $\mathscr{E}[\widetilde{\mathscr{D}}/\widetilde{z}] \twoheadrightarrow_\beta \check{\mathscr{D}}$, we have

$$\text{(17)} \quad \widetilde{\mathscr{D}}''[\widetilde{\mathscr{C}}_{k+1}/\widetilde{u}_{k+1}] \twoheadrightarrow_\beta \begin{array}{c} \hat{v} : (G_i)_{i \in P^+} \to A_{\hat{l}+1}^l \to B \quad \begin{pmatrix} \Gamma_i'', ((u_j : A_j)^\circ)_{j=1}^{\hat{l}} \\ \mathscr{B}_i \\ G_i \end{pmatrix}_{i \in P^+} \\ \hline\hline \begin{array}{c} A_{\hat{l}+1}^l \to B \\ \hline\hline A_1^l \to B \end{array} \to I, u_1^{\hat{l}} \end{array} \to E$$

$\widetilde{\mathscr{D}}''$ must have the following form:

$$\widetilde{\mathscr{D}}'' = \begin{array}{c} \widetilde{u}_{k+1} : \widetilde{G}_1^{\widetilde{n}} \to A_{\widetilde{l}+1}^l \to B \quad \begin{pmatrix} \widetilde{\Gamma}_i, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}} \\ \widetilde{\mathscr{G}}_i \\ \widetilde{G}_i \end{pmatrix}_{i=1}^{\widetilde{n}} \\ \hline\hline \begin{array}{c} A_{\widetilde{l}+1}^l \to B \\ \hline\hline A_1^l \to B \end{array} \to I, u_1^{\widetilde{l}} \end{array} \to E$$

where $\widetilde{l} \leq \hat{l}$. Since $\widetilde{\mathscr{D}}$ satisfies condition (I3) of Definition 16, each $\widetilde{\mathscr{G}}_i$ must, too.

We show that $\widetilde{l} = \hat{l}$. Suppose $\widetilde{l} < \hat{l}$. Then $\widetilde{\mathscr{C}_{k+1}}$ must have the following form:

$$\widetilde{\mathscr{C}_{k+1}} = \cfrac{\hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B \quad \begin{pmatrix} \Xi_i \\ \mathscr{E}''_i \\ G_{s_i} \end{pmatrix}^{|P^+|}_{i=1}}{\cfrac{\cfrac{A^l_{\hat{l}+1} \to B}{A^l_{\widetilde{l}+1} \to B} \to I, u^{\hat{l}}_{\widetilde{l}+1}}{\widetilde{G^n_1} \to A^l_{\widetilde{l}+1} \to B} \to I, \widetilde{v}^{\widetilde{n}}_1} \to E$$

where $(s_i)^{|P^+|}_{i=1}$ lists the elements of P^+ in increasing order and

$$\bigcup_{i=1}^{|P^+|} \Xi_i = (\widetilde{v}_i : \widetilde{G}_i)^{\widetilde{n}}_{i=1}, (u_j : A_j)^{\hat{l}}_{j=\widetilde{l}+1}.$$

Since $\widetilde{\mathscr{C}_{k+1}}$ satisfies condition (I3) of Definition 16, each $\mathscr{E}''_i$ must satisfy condition (I4) of Definition 16 (with respect to $(\Xi_i; \varnothing)$). By (17), it follows that, if $\Xi_i = (\widetilde{v}_j : \widetilde{G}_j)_{j \in J_1}, (u_j : A_j)_{j \in J_2}$, then $(\widetilde{\mathscr{G}_j})_{j \in J_1}, (u_j : A_j)_{j \in J_2}$ is an $\varnothing$-interpolant to $\mathscr{B}_{s_i}$ via $\mathscr{E}''_i$. Since $\mathscr{B}_{s_i}$ is connected by part 1 of Claim A, Lemma 32 implies that $|\Xi_i| = 1$ for each $i = 1, \dots, |P^+|$. Since $\hat{l} \geq \widetilde{l} + 1$, there is an i such that $u_{\hat{l}} : A_{\hat{l}} = \Xi_i$. Since $u_{\hat{l}} : A_{\hat{l}}$ is an $\varnothing$-interpolant to $\mathscr{B}_{s_i}$ via $\mathscr{E}''_i$, we see that $\mathscr{B}_{s_i} = \mathscr{E}''_i$ and every maximal path in $\mathscr{B}_i$ that starts inside its only assumption ends inside its endformula. By part 3 of Claim A, it follows that $\mathscr{B}_{s_i} = u_{\hat{l}} : A_{\hat{l}}$. Now take any $h \neq i$. Since $\mathscr{B}_{s_i} \neq \mathscr{B}_{s_h}$, the above argument shows that $u_{\hat{l}} : A_{\hat{l}} \notin \mathrm{Ass}(\mathscr{B}_{s_h})$, which contradicts the definition of $\hat{l}$.

We have shown that $\widetilde{l} = \hat{l}$. Now $\widetilde{\mathscr{C}_{k+1}}$ must have the following form:

$$\widetilde{\mathscr{C}_{k+1}} = \cfrac{\hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B \quad \begin{pmatrix} \Xi_i \\ \mathscr{E}''_i \\ G_{s_i} \end{pmatrix}^{t}_{i=1}}{\cfrac{\widetilde{G^n_{\widetilde{p}+1}} \to A^l_{\hat{l}+1} \to B}{\widetilde{G^n_1} \to A^l_{\hat{l}+1} \to B} \to I, \widetilde{v}^{\widetilde{p}}_1} \to E$$

where $t \leq |P^+|$, $\widetilde{p} \leq \widetilde{n}$, $\widetilde{G^n_{\widetilde{p}+1}} = (G_{s_i})^{|P^+|}_{i=t+1}$, and

$$\bigcup_{i=1}^{t} \Xi_i = (\widetilde{v}_i : \widetilde{G}_i)^{\widetilde{p}}_{i=1}.$$

Again, it is easy to see that we must have $|\Xi_i| = 1$, and if $\Xi_i = \widetilde{v}_j : \widetilde{G}_j$, then $\widetilde{\mathscr{G}_j}$ is an $\varnothing$-interpolant to $\mathscr{B}_{s_i}$ via $\mathscr{E}''_i$, which, by the induction hypothesis, implies that $\mathscr{B}_{s_i}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{G}_j}$. Since, by part 2 of Lemma 39, $\mathscr{B}_i$ is not an $\varnothing$-interpolant to $\mathscr{B}_j$ if $i < j$, it follows that $\Xi_i \cap \Xi_j = \varnothing$ if $i \neq j$. Therefore, $t = \widetilde{p}$ and $|P^+| = \widetilde{n}$. By (17), $\widetilde{\mathscr{G}_i} = \mathscr{B}_{s_i}$ for $t + 1 \leq i \leq |P^+|$. Thus, there is a bijection ρ from $\{1, \dots, |P^+|\}$

to P^+ such that

$$\widetilde{\mathscr{D}}'' = \cfrac{\widetilde{u}_{k+1} : \widetilde{G}_1^{|P^+|} \to A^l_{\hat{l}+1} \to B \qquad \begin{pmatrix} \Gamma''_{\rho(i)}, ((u_j : A_j)^\circ)_{j=1}^{\hat{l}} \\ \widetilde{\mathscr{G}_i} \\ \widetilde{G}_i \end{pmatrix}_{i=1}^{|P^+|}}{\cfrac{A^l_{\hat{l}+1} \to B}{A^l_1 \to B} \to I, u_1^{\hat{l}}} \to E$$

where for $i = 1, \ldots, |P^+|$, $\widetilde{\mathscr{G}_i}$ is an $\varnothing$-interpolant to $\mathscr{B}_{\rho(i)}$. We have shown that $\widetilde{\mathscr{D}}''$ has the required form. This proves the "only if" direction of the above statement.

Now suppose that $\widetilde{\mathscr{D}}$ is the normal form of (16). We will produce an auxiliary deduction $\mathscr{M}$ for $\widetilde{\mathscr{D}}, \check{\mathscr{D}}$ that is long for $\widetilde{\mathscr{D}}$, thereby proving the "if" direction of the above statement, and moreover prove that $\check{\mathscr{D}}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}$. By the induction hypothesis, let $\mathscr{M}' : \widetilde{w}_1 : \widetilde{C}_1^k \to (A_1^l \to B) \to C \Rightarrow C_1^k \to (A_1^l \to B) \to C$ be an auxiliary deduction for $\widetilde{\mathscr{D}}'_1, \mathscr{D}'_1$ that is long for $\widetilde{\mathscr{D}}'_1$, and for $i = 1, \ldots, |P^+|$, let $\mathscr{N}_i : \widetilde{v}_i : \widetilde{G}_i \Rightarrow G_{\rho(i)}$ be an auxiliary deduction for $\widetilde{\mathscr{G}_i}, \mathscr{B}_{\rho(i)}$ that is long for $\widetilde{\mathscr{G}_i}$. By part 2 of Lemma 44 and part 2 of Claim A, wee can see that $\mathscr{M}'$ must be of the following form:

$$\mathscr{M}' = \cfrac{\widetilde{w}_1 : \widetilde{C}_1^{\hat{k}} \to C^k_{\hat{k}+1} \to (A_1^l \to B) \to C \qquad \begin{pmatrix} \hat{u}_{\sigma(i)} : C_{\sigma(i)} \\ \widetilde{\mathscr{C}_i} \\ \widetilde{C}_i \end{pmatrix}_{i=1}^{\hat{k}}}{\cfrac{C^k_{\hat{k}+1} \to (A_1^l \to B) \to C}{C_1^k \to (A_1^l \to B) \to C} \to I, \hat{u}_1^{\hat{k}}} \to E$$

where $\hat{k}$ is as in (15), $\widetilde{C}^k_{\hat{k}+1} = C^k_{\hat{k}+1}$, and σ is a permutation of $\{1, \ldots, \hat{k}\}$. Let

$$\widetilde{\mathscr{C}_i} = \hat{u}_i : C_i \quad \text{for } i = \hat{k} + 1, \ldots, k,$$

$$\widetilde{\mathscr{C}}_{k+1} = \cfrac{\hat{v} : (G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B \qquad \begin{pmatrix} \widetilde{v}_{\rho^{-1}(i)} : \widetilde{G}_{\rho^{-1}(i)} \\ \mathscr{N}_{\rho^{-1}(i)} \\ G_i \end{pmatrix}_{i \in P^+}}{\cfrac{A^l_{\hat{l}+1} \to B}{\widetilde{G}_1^{|P^+|} \to A^l_{\hat{l}+1} \to B} \to I, (\widetilde{v}_i)_{i=1}^{|P^+|}} \to E$$

(Recall that $\widetilde{C}_{k+1} = \widetilde{G}_1^{|P^+|} \to A^l_{\hat{l}+1} \to B$.) Let $\mathscr{M} : \widetilde{z} : (\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C \Rightarrow C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C$ be the following deduction:

$$\mathscr{M} = \cfrac{\cfrac{\widetilde{z} : (\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C \qquad (\widetilde{\mathscr{C}}_{\pi(i)})_{i=1}^{k+1}}{C} \to E}{C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C} \to I, \hat{u}_1^k, \hat{v}$$

Then it is easy to see that $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ via $\mathscr{M}$. To prove that $\mathscr{M}$ is long for $\widetilde{\mathscr{D}}$, we use the induction hypothesis that $\mathscr{M}'$ and $\mathscr{N}_i$ are long for $\widetilde{\mathscr{D}}'$ and $\widetilde{\mathscr{G}_i}$, respectively. We leave the tedious but straightforward proof to the reader.

It remains to prove that $\breve{\mathscr{D}}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}$. By the induction hypothesis, $\mathscr{D}_1'$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_1'$ via some $\breve{\mathscr{M}}' : w_1 : C_1^k \to (A_1^l \to B) \to C \Rightarrow \widetilde{C}_1^k \to (A_1^l \to B) \to C$, and for $i = 1, \ldots, |P^+|$, $\mathscr{B}_{\rho(i)}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{G}}_i$ via some $\breve{\mathscr{N}}_i$: $v_{\rho(i)} : G_{\rho(i)} \Rightarrow \widetilde{G}_i$. By part 1 of Lemma 44, we may assume that $\breve{\mathscr{M}}'$ ends in $\hat{k}$ applications of $\to I$. Since $\mathscr{M}'[\breve{\mathscr{M}}'[\mathscr{D}_1'/w_i]/\widetilde{w}_1] \twoheadrightarrow_\beta \mathscr{D}_1'$, part 2 of Claim A implies that $\breve{\mathscr{M}}'$ must be of the following form:

$$\breve{\mathscr{M}}' = \dfrac{\dfrac{w_1 : C_1^k \to (A_1^l \to B) \to C \qquad \begin{pmatrix} \widetilde{u}_{\tau(i)} : \widetilde{C}_{\tau(i)} \\ \breve{\mathscr{C}}_i \\ C_i \end{pmatrix}_{i=1}^{\hat{k}}}{C_{\hat{k}+1}^k \to (A_1^l \to B) \to C} \to E}{\widetilde{C}_1^{\hat{k}} \to C_{\hat{k}+1}^k \to (A_1^l \to B) \to C} \to I, \widetilde{u}_1^{\hat{k}}$$

where τ is a permutation of $\{1, \ldots, \hat{k}\}$. Let

$$\breve{\mathscr{C}}_i = \widetilde{u}_i : C_i \quad \text{for } i = \hat{k} + 1, \ldots, k,$$

$$\breve{\mathscr{C}}_{k+1} = \dfrac{\dfrac{\widetilde{u}_{k+1} : \widetilde{G}_1^{|P^+|} \to A_{\hat{l}+1}^l \to B \qquad \begin{pmatrix} v_{\rho(i)} : G_{\rho(i)} \\ \breve{\mathscr{N}}_i \\ \widetilde{G}_i \end{pmatrix}_{i=1}^{|P^+|}}{A_{\hat{l}+1}^l \to B} \to E}{(G_i)_{i \in P^+} \to A_{\hat{l}+1}^l \to B} \to I, (v_i)_{i \in P^+}$$

Then let $\breve{\mathscr{M}} : \breve{z} : C_1^k \to ((G_i)_{i \in P^+} \to A_{\hat{l}+1}^l \to B) \to C \Rightarrow (\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C$ be the following deduction:

$$\breve{\mathscr{M}} = \dfrac{\dfrac{\breve{z} : C_1^k \to ((G_i)_{i \in P^+} \to A_{\hat{l}+1}^l \to B) \to C \qquad \breve{\mathscr{C}}_1^{k+1}}{C} \to E}{(\widetilde{C}_{\pi(i)})_{i=1}^{k+1} \to C} \to I, (\widetilde{u}_{\pi(i)})_{i=1}^{k+1}$$

It is easy to see that $\breve{\mathscr{D}}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}$ via $\breve{\mathscr{M}}$.

Case 2. $\breve{\mathscr{D}}$ is first constructed in Case 2.2.2 of the Induction Step of the new method, i.e., $\breve{\mathscr{D}}$ is the normal form of (13), repeated below:

(13)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma_1' \\ \mathscr{D}_1' \\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \quad (\hat{u}_j : C_j)_{j=1}^k}{(A_1^l \to B) \to C} \to E \qquad \dfrac{\dfrac{\begin{matrix}\Gamma_1'', ((u_j : A_j)^\circ)_{j=1}^l \\ \mathscr{B}_1 \\ H_1^q \to B\end{matrix} \qquad \left(\dfrac{\hat{v}_i : (G_j)_{j \in P_i^+} \to H_i \qquad \begin{pmatrix} \Gamma_j'', ((u_h : A_h)^\circ)_{h=1}^l \\ \mathscr{B}_j \\ G_j \end{pmatrix}_{j \in P_i^+}}{H_i} \to E\right)_{i=1}^q}{B} \to E}{A_1^l \to B} \to I, u_1^l}{C} \to E}{C_1^k \to ((G_j)_{j \in P_i^+} \to H_i)_{i=1}^q \to C} \to I, \hat{u}_1^k, \hat{v}_1^q$$

Let $\hat{k}$ and $\mathscr{D}_1'^-$ be as in (15). Let $\breve{\mathscr{B}}$ be the normal form of the subdeduction of (13) whose endformula is B. Let $\breve{\mathscr{H}}_i$ be the maximal subdeduction of $\breve{\mathscr{B}}$ which does not end in $\to I$ and whose main branch leads to $\hat{v}_i : (G_j)_{j \in P_i^+} \to H_i$. (By part 2 of

Claim A, $\check{\mathscr{H}}_i$ is unique.) $\check{\mathscr{H}}_i$ is of the form

$$\check{\mathscr{H}}_i = \dfrac{\hat{v}_i : (G_j)_{j\in P_i^+} \to (H_{i,j})_{j=1}^{r_i} \to H_{i,0} \qquad \begin{pmatrix}\Gamma''_j, ((u_h : A_h)^\circ)_{h=1}^{l} \\ \mathscr{B}_j \\ G_j\end{pmatrix}_{j\in P_i^+} \qquad \begin{pmatrix}\Psi_{i,j} \\ \check{\mathscr{H}}_{i,j} \\ H_{i,j}\end{pmatrix}_{j=1}^{r_i}}{H_{i,0}} \to E$$

where

$$H_i = (H_{i,j})_{j=1}^{r_i} \to H_{i,0}.$$

By part 2 of Claim A, $\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q$ do not overlap with each other and each $\check{\mathscr{H}}_{i,j}$ is a deduction in $\mathbb{D}$. If we write $\mathbf{B}[\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q]$ for $\check{\mathscr{B}}$, then

(18)

$$\check{\mathscr{D}} = \dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'_1, (\hat{u}_i : C_i)_{i=1}^{\hat{k}} \\ \mathscr{D}'^{-}_1 \\ C^k_{\hat{k}+1} \to (A^l_1 \to B) \to C\end{array} \qquad (\hat{u}_i : C_i)^k_{i=\hat{k}+1}}{(A^l_1 \to B) \to C} \to E \qquad \dfrac{\begin{array}{c}\Gamma''_1, \bigcup_{j\in P^+} \Gamma''_j, ((u_j : A_j)^\circ)^l_{j=1}, (\hat{v}_i : (G_j)_{j\in P_i^+} \to H_i)^q_{i=1} \\ \mathbf{B}[\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q] \\ B\end{array}}{A^l_1 \to B} \to I, u^l_1}{C} \to E}{C^k_1 \to ((G_j)_{j\in P_i^+} \to H_i)^q_{i=1} \to C} \to I, \hat{u}^k_1, \hat{v}^q_1$$

Let $(s_{i,j})_{j=1}^{|P_i^+|}$ list the elements of P_i^+ in increasing order. We can show that $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ if and only if $\widetilde{\mathscr{D}}$ is the normal form of a deduction of the form

$$(19) \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'_1 \\ \widetilde{\mathscr{D}}'_1 \\ \widetilde{C}^k_1 \to (A^l_1 \to B) \to C\end{array} \qquad (\widetilde{u}_i : \widetilde{C}_i)^k_{i=1}}{(A^l_1 \to B) \to C} \to E \qquad \dfrac{\begin{array}{c}\Gamma''_1, \bigcup_{j\in P^+} \Gamma''_j, ((u_j : A_j)^\circ)^l_{j=1}, (\widetilde{u}_i : \widetilde{C}_i)^{k+q}_{i=k+1} \\ \mathbf{B}[\widetilde{\mathscr{H}}_1, \dots, \widetilde{\mathscr{H}}_q] \\ B\end{array}}{A^l_1 \to B} \to I, u^l_1}{C} \to E}{(\widetilde{C}_{\pi(i)})^{k+q}_{i=1} \to C} \to I, (\widetilde{u}_{\pi(i)})^{k+q}_{i=1}$$

where π is a permutation of $\{1, \dots, k+q\}$, $\widetilde{\mathscr{D}}'_1$ is an $\varnothing$-interpolant to $\mathscr{D}'_1$, $\mathbf{B}[\widetilde{\mathscr{H}}_1, \dots, \widetilde{\mathscr{H}}_q]$ is the result of replacing $\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q$ in $\mathbf{B}[\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q]$ by $\widetilde{\mathscr{H}}_1, \dots, \widetilde{\mathscr{H}}_q$, respectively, and for $i = 1, \dots, q$,

$$\widetilde{C}_{k+i} = (\widetilde{G}_{i,j})_{j=1}^{|P_i^+|+r_i} \to H_{i,0},$$

$$\widetilde{\mathscr{H}}_i = \dfrac{\widetilde{u}_{k+i} : (\widetilde{G}_{i,j})_{j=1}^{|P_i^+|+r_i} \to H_{i,0} \qquad \begin{pmatrix}\Phi_{i,j} \\ \widetilde{\mathscr{G}}_{i,j} \\ \widetilde{G}_{i,j}\end{pmatrix}_{j=1}^{|P_i^+|+r_i}}{H_{i,0}} \to E$$

and there is a permutation ρ_i of $\{1, \dots, |P_i^+| + r_i\}$ such that

$$\widetilde{\mathscr{G}}_{i,j} \text{ is an } \varnothing\text{-interpolant to } \begin{cases}\mathscr{B}_{s_{i,\rho_i(j)}} & \text{if } 1 \le \rho_i(j) \le |P_i^+|, \\ \check{\mathscr{H}}_{i,\rho_i(j)-|P_i^+|} & \text{if } |P_i^+| + 1 \le \rho_i(j) \le |P_i^+| + r_i.\end{cases}$$

We first prove the "only if" direction of this statement. Suppose that $\widetilde{\mathscr{D}}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}$ via $\mathscr{E}$. Since $\check{\mathscr{D}}$ is connected by part 1 of Claim A, Lemma 32 implies that $\mathscr{E}$ can have only one assumption. By part 1 of Lemma 44, we may assume that $\mathscr{E}$ is of the following form:

$$\mathscr{E} = \cfrac{\cfrac{\widetilde{z} : \widetilde{F}_1^{\widetilde{k}} \to C \qquad \begin{pmatrix}\Theta_i \\ \widetilde{\mathscr{F}_i} \\ \widetilde{F}_i\end{pmatrix}_{i=1}^{\widetilde{k}}}{C} \,{\to}E}{C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C} \,{\to}I, \hat{u}_1^k, \hat{v}_1^q$$

where

$$\Theta_1 \cup \cdots \cup \Theta_{\widetilde{k}} = (\hat{u}_i : C_i)_{i=1}^k, (\hat{v}_i : (G_j)_{j\in P_i^+} \to H_i)_{i=1}^q.$$

Since $\mathscr{E}$ satisfies condition (I4) of Definition 16 (with respect to $(\widetilde{z} : \widetilde{F}_1^{\widetilde{k}} \to C; \varnothing)$), each $\widetilde{\mathscr{F}_i}$ must satisfy condition (I3) of Definition 16. This implies that $\Theta_i \neq \varnothing$ for each i. Moreover, by part 2 of Claim A and Lemma 19, it is not difficult to see that

$$|\Theta_i| = 1 \quad \text{for each } i,$$
$$\Theta_i \cap \Theta_j = \varnothing \quad \text{if } i \neq j.$$

So $\widetilde{k} = k+q$. By part 3 of Lemma 40, the main branch of $\widetilde{\mathscr{D}}$ leads to an assumption belonging to Γ_1'. Note that $\widetilde{F}_{\widetilde{k}}$ cannot be the subdeduction of (18) whose endformula is $A_1^l \to B$. For, if that subdeduction satisfies condition (I3) of Definition 16, its main branch must lead to an assumption belonging to $\Gamma_1'' \neq \varnothing$. Therefore, $\widetilde{\mathscr{D}}$ must be of the following form:

$$\widetilde{\mathscr{D}} = \cfrac{\cfrac{\begin{matrix}\Gamma_1', (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^k \\ \widetilde{\mathscr{D}}' \\ (A_1^l \to B) \to C\end{matrix} \qquad \begin{matrix}\Gamma_1'' \cup \bigcup_{j\in P^+} \Gamma_j'', (\widetilde{u}_i : \widetilde{C}_i)_{i=k+1}^{k+q} \\ \widetilde{\mathscr{D}}'' \\ A_1^l \to B\end{matrix}}{C} \,{\to}E}{(\widetilde{C}_{\pi(i)})_{i=1}^{k+q} \to C} \,{\to}I, (\widetilde{u}_{\pi(i)})_{i=1}^{k+q}$$

where π is a permutation of $\{1, \ldots, k+q\}$, $\widetilde{F}_i = \widetilde{C}_{\pi(i)}$, and

$$\Theta_i = \begin{cases} \hat{u}_{\pi(i)} : C_{\pi(i)} & \text{if } 1 \leq \pi(i) \leq k, \\ \hat{v}_{\pi(i)-k} : (G_j)_{j\in P_{\pi(i)-k}^+} \to H_{\pi(i)-k} & \text{if } k+1 \leq \pi(i) \leq k+q. \end{cases}$$

We have to show that $\widetilde{\mathscr{D}}'$ and $\widetilde{\mathscr{D}}''$ have the required form. Let $\widetilde{\mathscr{C}_i} = \widetilde{\mathscr{F}}_{\pi^{-1}(i)}$, so that

$$\widetilde{\mathscr{C}_i} \colon \hat{u}_i : C_i \Rightarrow \widetilde{C}_i \quad \text{for } i = 1, \ldots, k,$$
$$\widetilde{\mathscr{C}}_{k+i} \colon \hat{v}_i : (G_j)_{j\in P_i^+} \to H_i \Rightarrow \widetilde{C}_{k+i} \quad \text{for } i = 1, \ldots, q.$$

Exactly as in Case 1, we can show

$$\cfrac{\begin{matrix}\Gamma_1' \\ \widetilde{\mathscr{D}}_1' \\ \widetilde{C}_1^k \to (A_1^l \to B) \to C\end{matrix} \qquad (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^k}{(A_1^l \to B) \to C} \,{\to}E \quad \twoheadrightarrow_\beta \quad \widetilde{\mathscr{D}}'$$

where $\widetilde{\mathscr{D}}_1'$ is an $\varnothing$-interpolant to $\mathscr{D}_1'$.

We turn to $\widetilde{\mathscr{D}}''$. Since $\mathscr{E}[\widetilde{\mathscr{D}}/\widetilde{z}] \twoheadrightarrow_\beta \check{\mathscr{D}}$, the main branch of $\widetilde{\mathscr{D}}''$ must lead to an assumption belonging to Γ'' and $\widetilde{\mathscr{D}}''$ must be of the following form

$$\widetilde{\mathscr{D}}'' = \cfrac{\cfrac{\begin{matrix}\Gamma_1'' \cup \bigcup_{j\in P^+} \Gamma_j'', ((u_j : A_j)^\circ)_{j=1}^l, (\widetilde{u}_i : \widetilde{C}_i)_{i=k+1}^{k+q} \\ \widetilde{\mathscr{B}} \\ B\end{matrix}}{A_1^l \to B} \to I, u_1^l}{}$$

We have

$$\widetilde{\mathscr{B}}[(\widetilde{\mathscr{C}}_i/\widetilde{u}_i)_{i=k+1}^{k+q}] \twoheadrightarrow_\beta \mathbf{B}[\check{\mathscr{H}}_1, \dots, \check{\mathscr{H}}_q].$$

Since $\widetilde{\mathscr{C}}_{k+i}$ satisfies condition (I3) of Definition 16, the main branch of $\widetilde{\mathscr{C}}_{k+i}$ leads to $\hat{v}_i : (G_j)_{j\in P_i^+} \to (H_{i,j})_{j=1}^{r_i} \to H_{i,0}$. Then it is not difficult to see that

$$\widetilde{\mathscr{B}} = \mathbf{B}[\widetilde{\mathscr{H}_1}, \dots, \widetilde{\mathscr{H}_q}],$$

where, for each $i = 1, \dots, q$, $\widetilde{\mathscr{H}_i}$ is of the form

$$\widetilde{\mathscr{H}_i} = \cfrac{\widetilde{u}_{k+i} : (\widetilde{G}_{i,j})_{j=1}^{\widetilde{n}_i} \to H_{i,0} \qquad \begin{pmatrix}\Phi_{i,j} \\ \widetilde{\mathscr{G}}_{i,j} \\ \widetilde{G}_{i,j}\end{pmatrix}_{j=1}^{\widetilde{n}_i}}{H_{i,0}} \to E$$

and

$$\widetilde{\mathscr{H}_i}[\widetilde{\mathscr{C}}_{k+i}/\widetilde{u}_{k+i}] \twoheadrightarrow_\beta \check{\mathscr{H}}_i.$$

Since $\widetilde{\mathscr{D}}$ satisfies condition (I3) of Definition 16, each $\widetilde{\mathscr{G}}_{i,j}$ does, too. For $i = 1, \dots, q$, $\widetilde{\mathscr{C}}_{k+i}$ must have the following form:

$$\widetilde{\mathscr{C}}_{k+i} = \cfrac{\hat{v}_i : (G_{s_{i,j}})_{j=1}^{|P_i^+|} \to (H_{i,j})_{j=1}^{r_i} \to H_{i,0} \qquad \begin{pmatrix}\Xi_{i,j} \\ \mathscr{E}''_{i,j} \\ G_{i,j}\end{pmatrix}_{j=1}^{t_i}}{\cfrac{(\widetilde{G}_{i,j})_{j=\widetilde{p}_i+1}^{\widetilde{n}_i} \to H_{i,0}}{(\widetilde{G}_{i,j})_{j=1}^{\widetilde{n}_i} \to H_{i,0}} \to I, (\widetilde{v}_{i,j})_{j=1}^{\widetilde{p}_i}} \to E$$

where $t_i \le |P_i^+| + r_i$, $\widetilde{p}_i \le \widetilde{n}_i$, and

$$(G_{i,j})_{j=1}^{t_i}, (\widetilde{G}_{i,j})_{j=\widetilde{p}_i+1}^{\widetilde{n}_i} = (G_{s_{i,j}})_{j=1}^{|P_i^+|}, (H_{i,j})_{j=1}^{r_i},$$

$$\bigcup_{j=1}^{t_i} \Xi_{i,j} = (\widetilde{v}_{i,j} : \widetilde{G}_{i,j})_{j=1}^{\widetilde{p}_i}.$$

Since $\widetilde{\mathscr{C}}_{k+i}$ satisfies condition (I3) of Definition 16, each $\mathscr{E}''_{i,j}$ must satisfy condition (I4) of Definition 16 with respect to $(\Xi_{i,j}; \varnothing)$. For $h = 1, \dots, t_i$, let $J_{i,h} = \{ j \mid \widetilde{v}_{i,j} : \widetilde{G}_{i,j} \in \Xi_{i,j} \}$. Since $\widetilde{\mathscr{H}_i}[\widetilde{\mathscr{C}}_{k+i}/\widetilde{u}_{k+i}] \twoheadrightarrow_\beta \check{\mathscr{H}}_i$, we have

$$(20) \qquad (\widetilde{\mathscr{G}}_{i,j})_{j\in J_{i,h}} \text{ is an } \varnothing\text{-interpolant to } \begin{cases} \mathscr{B}_{s_{i,h}} \text{ via } \mathscr{E}''_{i,h} & \text{if } h \le |P_i^+|, \\ \check{\mathscr{H}}_{i,h-|P_i^+|} \text{ via } \mathscr{E}''_{i,h} & \text{if } h > |P_i^+|. \end{cases}$$

Since $\mathscr{B}_{s_{i,h}}$ and $\check{\mathscr{H}}_{i,h-|P_i^+|}$ are connected by part 1 of Claim A, Lemma 32 implies that $|J_{i,h}| = 1$. By part 2 of Claim A and the induction hypothesis, we can see that $J_{i,h} \cap J_{i,h'} = \emptyset$ if $h \neq h'$. Therefore, $t_i = \widetilde{p}_i$ and $\widetilde{n}_i = |P_i^+| + r_i$. For $h = t_i + 1, \ldots, |P_i^+| + r_i$, define

$$\widetilde{\mathscr{G}}_{i,h} = \begin{cases} \mathscr{B}_{s_{i,h}} & \text{if } h \leq |P_i^+|, \\ \check{\mathscr{H}}_{i,h-|P_i^+|} & \text{if } h > |P_i^+|. \end{cases} \tag{21}$$

Combining (20) and (21), we conclude that there is a permutation ρ_i of $\{1, \ldots, |P_i^+| + r_i\}$ such that

$$\widetilde{\mathscr{G}}_{i,j} \text{ is an } \emptyset\text{-interpolant to } \begin{cases} \mathscr{B}_{s_{i,\rho_i(j)}} & \text{if } 1 \leq \rho_i(j) \leq |P^+|, \\ \check{\mathscr{H}}_{i,\rho_i(j)-|P_i^+|} & \text{if } |P_i^+| + 1 \leq \rho_i(j) \leq |P_i^+| + r_i. \end{cases}$$

We have shown that $\widetilde{\mathscr{D}}''$ has the required form. This proves the “only if” direction of the above statement.

Conversely, suppose that $\widetilde{\mathscr{D}}$ is the normal form of (19). By the induction hypothesis, let $\mathscr{M}' : \widetilde{w}_1 : \widetilde{C}_1^k \to (A_1^l \to B) \to C \Rightarrow C_1^k \to (A_1^l \to B) \to C$ be an auxiliary deduction for $\widetilde{\mathscr{D}}_1', \mathscr{D}_1'$ that is long for $\widetilde{\mathscr{D}}_1'$, and for $i = 1, \ldots, q$, let $\mathscr{N}_{i,j} : \widetilde{v}_{i,j} : \widetilde{G}_{i,j} \Rightarrow G_{s_{i,\rho_i(j)}}$ be an auxiliary deduction for $\widetilde{\mathscr{G}}_{i,j}, \mathscr{B}_{s_{i,\rho_i(j)}}$ (in case $1 \leq \rho_i(j) \leq |P_i^+|$) or an auxiliary deduction for $\widetilde{\mathscr{G}}_{i,j}, \check{\mathscr{H}}_{i,\rho_i(j)-|P_i^+|}$ (in case $|P_i^+| + 1 \leq \rho_i(j) \leq |P_i^+| + r_i$) that is long for $\widetilde{\mathscr{G}}_{i,j}$. As in the previous case, we can see that $\mathscr{M}'$ must be of the following form:

$$\mathscr{M}' = \cfrac{\cfrac{\widetilde{w}_1 : \widetilde{C}_1^{\hat{k}} \to C_{\hat{k}+1}^k \to (A_1^l \to B) \to C \quad \begin{pmatrix} \hat{u}_{\sigma(i)} : C_{\sigma(i)} \\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i \end{pmatrix}_{i=1}^{\hat{k}}}{C_{\hat{k}+1}^k \to (A_1^l \to B) \to C} \to E}{C_1^k \to (A_1^l \to B) \to C} \to I, \hat{u}_1^{\hat{k}}$$

where $\widetilde{C}_{\hat{k}+1}^k = C_{\hat{k}+1}^k$ and σ is a permutation of $\{1, \ldots, \hat{k}\}$. Let

$$\widetilde{\mathscr{C}}_i = \hat{u}_i : C_i \quad \text{for } i = \hat{k} + 1, \ldots, k,$$

$$\widetilde{\mathscr{C}}_{k+i} = \cfrac{\cfrac{\hat{v}_i : (G_j)_{j \in P_i^+} \to (H_{i,j})_{j=1}^{r_i} \to H_{i,0} \quad \begin{pmatrix} \widetilde{v}_{i,\rho_i^{-1}(j)} : \widetilde{G}_{i,\rho_i^{-1}(j)} \\ \mathscr{N}_{i,\rho_i^{-1}(j)} \\ G_{s_{i,j}} \end{pmatrix}_{j=1}^{|P_i^+|} \quad \begin{pmatrix} \widetilde{v}_{i,\rho_i^{-1}(j)} : \widetilde{G}_{i,\rho_i^{-1}(j)} \\ \mathscr{N}_{i,\rho_i^{-1}(j)} \\ H_{i,j-|P_i^+|} \end{pmatrix}_{j=|P_i^+|+1}^{|P_i^+|+r_i}}{H_{i,0}} \to E}{(\widetilde{G}_{i,j})_{j=1}^{|P_i^+|+r_i} \to H_{i,0}} \to I, (\widetilde{v}_{i,j})_{j=1}^{|P_i^+|+r_i}$$

for $i = 1, \ldots, q$.

Then let $\mathscr{M} : \widetilde{z} : (\widetilde{C}_{\pi(i)})_{i=1}^{k+q} \to C \Rightarrow C_1^k \to ((G_j)_{j \in P_i^+} \to H_i)_{i=1}^q \to C$ be the following deduction:

$$\mathscr{M} = \cfrac{\cfrac{\widetilde{z} : (\widetilde{C}_{\pi(i)})_{i=1}^{k+q} \to C \quad (\widetilde{\mathscr{C}}_{\pi(i)})_{i=1}^{k+q}}{C} \to E}{C_1^k \to ((G_j)_{j \in P_i^+} \to H_i)_{i=1}^q \to C} \to I, \hat{u}_1^k, \hat{v}_1^q$$

Then it is easy to see that $\widetilde{\mathscr{D}}$ is an ∅-interpolant to $\check{\mathscr{D}}$ via $\mathscr{M}$. To prove that $\mathscr{M}$ is long for $\widetilde{\mathscr{D}}$, we use the induction hypothesis that $\mathscr{M}'$ and $\mathscr{N}_{i,j}$ are long for $\widetilde{\mathscr{D}}'_1$ and $\widetilde{\mathscr{G}}_{i,j}$, respectively. We leave the tedious but straightforward proof to the reader.

It remains to prove that $\check{\mathscr{D}}$ is an ∅-interpolant to $\widetilde{\mathscr{D}}$. By the induction hypothesis, $\mathscr{D}'_1$ is an ∅-interpolant to $\widetilde{\mathscr{D}}'_1$ via some $\check{\mathscr{M}}'\colon w_1 : C_1^k \to (A_1^l \to B) \to C \Rightarrow \widetilde{C}_1^k \to (A_1^l \to B) \to C$, and for $i = 1, \ldots, q$ and $j = 1, \ldots, |P_i^+| + r_i$, $\mathscr{B}_{\rho_i(j)}$ is an ∅-interpolant to $\widetilde{\mathscr{G}}_{i,j}$ via some $\check{\mathscr{N}}_{i,j}\colon v_{s_{i,\rho_i(j)}} : G_{s_{i,\rho_i(j)}} \Rightarrow \widetilde{G}_{i,j}$ if $1 \le \rho_i(j) \le |P_i^+|$ and $\check{\mathscr{H}}_{i,\rho_i(j)-|P_i^+|}$ is an ∅-interpolant to $\widetilde{\mathscr{G}}_{i,j}$ via some $\check{\mathscr{N}}_{i,j}\colon y_{i,\rho_i(j)-|P_i^+|} : H_{i,\rho_i(j)-|P_i^+|} \Rightarrow \widetilde{G}_{i,j}$ if $|P_i^+| + 1 \le \rho_i(j) \le |P_i^+| + r_i$. As in the previous case, we may assume that $\check{\mathscr{M}}'$ is of the following form:

$$\check{\mathscr{M}}' = \cfrac{\cfrac{w_1 : C_1^k \to (A_1^l \to B) \to C \qquad \begin{pmatrix} \widetilde{u}_{\tau(i)} : \widetilde{C}_{\tau(i)} \\ \check{\mathscr{C}}_i \\ C_i \end{pmatrix}_{i=1}^{\hat{k}}}{C^k_{\hat{k}+1} \to (A_1^l \to B) \to C}\,{\to}E}{\widetilde{C}_1^{\hat{k}} \to C^k_{\hat{k}+1} \to (A_1^l \to B) \to C}\,{\to}I, \widetilde{u}_1^{\hat{k}}$$

where τ is a permutation of $\{1, \ldots, \hat{k}\}$. Let

$$\check{\mathscr{C}}_i = \widetilde{u}_i : C_i \quad \text{for } i = \hat{k} + 1, \ldots, k,$$

$$\check{\mathscr{C}}_{k+i} = \cfrac{\widetilde{u}_{k+i} : (\widetilde{G}_{i,j})_{j=1}^{|P_i^+|+r_i} \to H_{i,0} \qquad \begin{pmatrix} (v_{s_{i,\rho_i(j)}} : G_{s_{i,\rho_i(j)}})^\circ, (y_{i,\rho_i(j)-|P_i^+|} : H_{i,\rho_i(j)-|P_i^+|})^\circ \\ \check{\mathscr{N}}_{i,j} \\ \widetilde{G}_{i,j} \end{pmatrix}_{j=1}^{|P_i^+|+r_i}}{\cfrac{H_{i,0}}{(G_j)_{j \in P_i^+} \to (H_{i,j})_{j=1}^{r_i} \to H_{i,0}}\,{\to}I, (v_j)_{j \in P_i^+}, (y_{i,j})_{j=1}^{r_i}}\,{\to}E$$

for $i = 1, \ldots, q$.

Then let $\check{\mathscr{M}}\colon \check{z} : C_1^k \to ((G_j)_{j \in P_i^+} \to H_i)_{i=1}^q \to C \Rightarrow (\widetilde{C}_{\pi(i)})_{i=1}^{k+q} \to C$ be the following deduction:

$$\check{\mathscr{M}} = \cfrac{\cfrac{\check{z} : C_1^k \to ((G_j)_{j \in P_i^+} \to H_i)_{i=1}^q \to C \qquad \check{\mathscr{C}}_1^{k+q}}{C}\,{\to}E}{(\widetilde{C}_{\pi(i)})_{i=1}^{k+q} \to C}\,{\to}I, (\widetilde{u}_{\pi(i)})_{i=1}^{k+q}$$

It is easy to see that $\check{\mathscr{D}}$ is an ∅-interpolant to $\widetilde{\mathscr{D}}$ via $\check{\mathscr{M}}$. This completes the proof of Claim B.

We have described an algorithm that, given an arbitrary normal deduction $\widetilde{\mathscr{D}}$ and a deduction $\check{\mathscr{D}}$ which is among the deductions $\check{\mathscr{D}}_1^{\check{m}}$ constructed during the course of the new method, determines whether $\widetilde{\mathscr{D}}$ is an ∅-interpolant to $\check{\mathscr{D}}$, and if so, computes a particular auxiliary deduction $\mathscr{M}$ for $\widetilde{\mathscr{D}}, \check{\mathscr{D}}$. We can use this algorithm to compute $\mu(i)$ and $\mathscr{M}_i$ used in the definition of the function prune. We will assume that $\mathscr{M}_i\colon z_{\mu(i)} : E_{\mu(i)} \Rightarrow \check{E}_i$ is the deduction returned by the above algorithm on input $\mathscr{D}_{\mu(i)}, \check{\mathscr{D}}_i$. In particular, for each i, we have the following:

(22) $\mathscr{D}_{\mu(i)}$ is an ∅-interpolant to $\check{\mathscr{D}}_i$ via $\mathscr{M}_i$.

(23) $\mathscr{M}_i$ is long for $\mathscr{D}_{\mu(i)}$.

Note that part 2 of Lemma 39 and part 2 of Claim B together imply that the interpolant $\mathscr{D}_1^m$ constructed by the new method satisfies the following property:

(24) If $i \neq j$, $\mathscr{D}_i$ is not an $\varnothing$-interpolant to $\mathscr{D}_j$.

From (22) and (24), we also get:

(25) If $\check{\mathscr{D}}_i$ is an $\varnothing$-interpolant to $\mathscr{D}_j$, then $j = \mu(i)$.

As a consequence of part 1 of Claim A and part 2 of Claim B, we know that $\mathscr{D}_1^m$ is a maximally strong interpolant to $\mathscr{D}$ in the sense that no interpolant to $\mathscr{D}$ is strictly stronger than it. This is still short of establishing that $\mathscr{D}_1^m$ is in fact a strongest interpolant, which we are now going to prove.

Claim C. Let $(\mathscr{D}_i \colon \Gamma_i \Rightarrow E_i)_{i=1}^m, \mathscr{D}_0 \colon (z_i : E_i)_{i=1}^m, \Delta \Rightarrow C$ be the deductions that the new method outputs when given deduction $\mathscr{D} \colon \Gamma, \Delta \Rightarrow C$ together with the partition $(\Gamma; \Delta)$ as input. Suppose that $(\widetilde{\mathscr{D}}_i \colon \widetilde{\Gamma}_i \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ is another interpolant to $\mathscr{D}$ with respect to the partition $(\Gamma; \Delta)$ via $\widetilde{\mathscr{D}}_0 \colon (\widetilde{z}_i : \widetilde{E}_i)_{i=1}^{\widetilde{m}}, \Delta \Rightarrow C$. Then one can find $\widetilde{m}$ subsets $S_1, \dots, S_{\widetilde{m}}$ of $\{1, \dots, m\}$ and $\widetilde{m}$ normal deductions $(\mathscr{E}_i \colon (z_j : E_j)_{j \in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ satisfying the following conditions:

1. $S_1 \cup \cdots \cup S_{\widetilde{m}} = \{1, \dots, m\}$;
2. For $i = 1, \dots, \widetilde{m}$, $(\mathscr{D}_j)_{j \in S_i}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i$;
3. $\widetilde{\mathscr{D}}_0[(\mathscr{E}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] \twoheadrightarrow_\beta \mathscr{D}_0$;
4. For each $i = 1, \dots, \widetilde{m}$ and for each $j \in S_i$, $\mathscr{E}_i$ is long for $\mathscr{D}_j$ with respect to $z_j : E_j$.

Note that conditions 1–2 simply say that $\mathscr{D}_1^m$ is stronger than $\widetilde{\mathscr{D}}_1^{\widetilde{m}}$.

We prove the claim by induction on $\mathscr{D}$, following mostly the description of the construction of $\mathscr{D}_1^m, \mathscr{D}_0$. It suffices to prove conditions 2–4, because condition 1 easily follows from condition 3.

Induction Basis. $\mathscr{D}$ is $x : C$.

Case 1. $\Gamma = \{x : C\}, \Delta = \varnothing$. We have $m = 1$, $\mathscr{D}_1 = \mathscr{D}$, and $\mathscr{D}_0 = z_1 : C$. By Lemma 19, $\widetilde{m} = 1$, and by Lemma 40, $\widetilde{\mathscr{D}}_0$ does not end in $\to I$, and the main branch of $\widetilde{\mathscr{D}}_0$ leads to $\widetilde{z}_1$. It follows that $\widetilde{\mathscr{D}}_0 = \widetilde{z}_1 : C$ and $\widetilde{\mathscr{D}}_1 = \mathscr{D}$. So the claim holds with $\mathscr{E}_1 = z_1 : C$.

Case 2. $\Gamma = \varnothing, \Delta = \{x : C\}$. We have $m = 0$ and $\mathscr{D}_0 = \mathscr{D}$. We must have $\widetilde{m} = 0$ and the claim holds trivially.

Induction Step.

Case 1. The last inference of $\mathscr{D}$ is $\to I$. $\mathscr{D}$ is of the form:

$$\mathscr{D} = \quad \dfrac{\begin{matrix}(y : A)^\circ, \Gamma, \Delta \\ \mathscr{D}' \\ B\end{matrix}}{A \to B} \to I, y$$

where $A \to B = C$. We have $\mathscr{D}_1^m = \mathscr{D}'^m_1$ and

$$\mathscr{D}_0 = \quad \dfrac{\begin{matrix}(z_i : E_i)_{i=1}^m, (y : A)^\circ, \Delta \\ \mathscr{D}_0' \\ B\end{matrix}}{A \to B} \to I, y$$

where ${\mathscr{D}'}_1^m, \mathscr{D}'_0$ is the output of the new method on input $\mathscr{D}'$, $(\Gamma; (y : A)^\circ, \Delta)$. We have two subcases to consider.

Case 1a. $\widetilde{\mathscr{D}}_0$ ends in $\to I$. Then $\widetilde{\mathscr{D}}_0$ is of the form

$$\widetilde{\mathscr{D}}_0 = \begin{array}{c} (\widetilde{z}_i : \widetilde{E}_i)_{i=1}^{\widetilde{m}}, (y : A)^\circ, \Delta \\ \widetilde{\mathscr{D}}'_0 \\ \dfrac{B}{A \to B} \to I, y \end{array}$$

Then it is easy to check that $\widetilde{\mathscr{D}}_1^{\widetilde{m}}$ is an interpolant to $\mathscr{D}'$ with respect to the partition $(\Gamma; (y : A)^\circ, \Delta)$ via $\widetilde{\mathscr{D}}'_0$. The induction hypothesis then gives sets $S_1^{\widetilde{m}}$ and deductions $\mathscr{E}_1^{\widetilde{m}}$ with the necessary properties.

Case 1b. $\widetilde{\mathscr{D}}_0$ does not end in $\to I$. Then $\widetilde{\mathscr{D}}_0$ must look like the following:

$$\widetilde{\mathscr{D}}_0 = \begin{array}{c} \qquad\qquad\qquad\qquad (\widetilde{z}_1 : \widetilde{E}_1)^\circ, (\widetilde{z}_i : \widetilde{E}_i)_{i=2}^{\widetilde{m}}, \Delta \\ \qquad\qquad\qquad\qquad \widetilde{\mathscr{C}}_1^{\widetilde{k}} \\ \dfrac{\widetilde{z}_1 : \widetilde{C}_1^{\widetilde{k}} \to A \to B \qquad\qquad \widetilde{C}_1^{\widetilde{k}}}{A \to B} \to E \end{array}$$

where $\widetilde{E}_1 = \widetilde{C}_1^{\widetilde{k}} \to A \to B$. $\widetilde{\mathscr{D}}_1$ must have the following form:

$$\widetilde{\mathscr{D}}_1 = \begin{array}{c} \widetilde{\Gamma}_1, (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^{\widetilde{k}}, y : A \\ \widetilde{\mathscr{D}}_1^- \\ \dfrac{B}{\widetilde{C}_1^{\widetilde{k}} \to A \to B} \to I, \widetilde{u}_1^{\widetilde{k}}, y \end{array}$$

Let

$$\widetilde{\mathscr{D}}'_0 = \begin{array}{c} (\widetilde{z}_i : \widetilde{E}_i)_{i=1}^{\widetilde{m}}, \Delta \\ \widetilde{\mathscr{D}}_0 \\ \dfrac{A \to B \qquad y : A}{B} \to E \end{array}$$

Then $\widetilde{\mathscr{D}}'_0$ is a normal deduction and it is easy to see that $\widetilde{\mathscr{D}}_1^{\widetilde{m}}$ is an interpolant to $\mathscr{D}'$ with respect to the partition $(\Gamma; y : A, \Delta)$ via $\widetilde{\mathscr{D}}'_0$. By the induction hypothesis, we have subsets $S_1^{\widetilde{m}}$ of $\{1, \dots, m\}$ and deductions $(\mathscr{E}_i : (z_j : E_j)_{j \in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ such that

(26) i. $S_1 \cup \cdots \cup S_{\widetilde{m}} = \{1, \dots, m\}$;

ii. for $i = 1, \dots, \widetilde{m}$, $(\mathscr{D}_j)_{j \in S_i}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i$;

iii. $\widetilde{\mathscr{D}}'_0[(\mathscr{E}_i / \widetilde{z}_i)_1^{\widetilde{m}}] \twoheadrightarrow_\beta \mathscr{D}'_0$;

iv. for each $i = 1, \dots, \widetilde{m}$ and for each $j \in S_i$, $\mathscr{E}_i$ is long for $\mathscr{D}_i$ with respect to $z_j : E_j$.

Only condition 3 remains to be proved. By (26.iv) and part 2 of Lemma 44, $\mathscr{E}_1$ must look as follows:

$$\mathscr{E}_1 = \begin{array}{c} (z_j : E_j)_{j \in S_1}, (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^{\widetilde{k}}, y : A \\ \mathscr{E}_1^- \\ \dfrac{B}{\widetilde{C}_1^{\widetilde{k}} \to A \to B} \to I, \widetilde{u}_1^{\widetilde{k}}, y \end{array}$$

Then it is not hard to see that $\widetilde{\mathscr{D}}_0[(\mathscr{E}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] \twoheadrightarrow_\beta \mathscr{D}_0$.

Case 2. The last inference of $\mathscr{D}$ is $\to E$. $\mathscr{D}$ is of the form

$$\mathscr{D} = \dfrac{\begin{array}{c}\Gamma', \Delta' \\ \mathscr{D}' \\ C'' \to C\end{array} \qquad \begin{array}{c}\Gamma'', \Delta'' \\ \mathscr{D}'' \\ C''\end{array}}{C} \to E$$

where $\Gamma' \cup \Gamma'' = \Gamma$ and $\Delta' \cup \Delta'' = \Delta$.

In each of the following subcases, we have $\mathscr{D}_1^m, \mathscr{D}_0 = \text{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. We let $\mu(i)$ and $\mathscr{M}_i$ be as in the definition of $\text{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$.

Case 2.1. The main branch of $\mathscr{D}'$ leads to an assumption belonging to Δ'. Then by Lemma 40, the main branch of $\widetilde{\mathscr{D}}_0$ leads to an assumption belonging to Δ', and $\widetilde{\mathscr{D}}_0$ must look like the following, where $M' \cup M'' = \{1, \dots, \widetilde{m}\}$:

$$\widetilde{\mathscr{D}}_0 = \dfrac{\begin{array}{c}(\widetilde{z}_i : \widetilde{E}_i)_{i \in M'}, \Delta' \\ \widetilde{\mathscr{D}}_0' \\ C'' \to C\end{array} \qquad \begin{array}{c}(\widetilde{z}_i : \widetilde{E}_i)_{i \in M''}, \Delta'' \\ \widetilde{\mathscr{D}}_0'' \\ C''\end{array}}{C} \to E \tag{27}$$

It is easy to see that $(\widetilde{\mathscr{D}}_i)_{i \in M'}$ is an interpolant to $\mathscr{D}'$ with respect to the partition $(\Gamma'; \Delta')$ via $\widetilde{\mathscr{D}}_0'$, and $(\widetilde{\mathscr{D}}_i)_{i \in M''}$ is an interpolant to $\mathscr{D}''$ with respect to the partition $(\Gamma''; \Delta'')$ via $\widetilde{\mathscr{D}}_0''$. Applying the induction hypothesis to $\mathscr{D}'$ and $\mathscr{D}''$, we obtain subsets $(S_i')_{i \in M'}$ of $\{1, \dots, n\}$, subsets $(S_i'')_{i \in M''}$ of $\{1, \dots, p\}$, and deductions $(\mathscr{E}_i' \colon (w_j : F_j)_{j \in S_i'} \Rightarrow \widetilde{E}_i)_{i \in M'}$, $(\mathscr{E}_i'' \colon (v_j : G_j)_{j \in S_i''} \Rightarrow \widetilde{E}_i)_{i \in M''}$ such that

(28)
- i. $\bigcup_{i \in M'} S_i' = \{1, \dots, n\}$;
- ii. for each $i \in M'$, $(\mathscr{D}_j')_{j \in S_i'}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i'$;
- iii. $\widetilde{\mathscr{D}}_0'[(\mathscr{E}_i'/\widetilde{z}_i)_{i \in M'}] \twoheadrightarrow_\beta \mathscr{D}_0'$;
- iv. for each $i \in M'$ and for each $j \in S_i'$, $\mathscr{E}_i'$ is long for $\mathscr{D}_j'$ with respect to $w_j : F_j$;

(29)
- i. $\bigcup_{i \in M''} S_i'' = \{1, \dots, p\}$;
- ii. for each $i \in M''$, $(\mathscr{D}_j'')_{j \in S_i''}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i''$;
- iii. $\widetilde{\mathscr{D}}_0''[(\mathscr{E}_i''/\widetilde{z}_i)_{i \in M''}] \twoheadrightarrow_\beta \mathscr{D}_0''$;
- iv. for each $i \in M''$ and for each $j \in S_i''$, $\mathscr{E}_i''$ is long for $\mathscr{D}_j''$ with respect to $v_j : G_j$.

The output $\mathscr{D}_1^m, \mathscr{D}_0$ of the new method is the result of applying the pruning procedure to $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow \check{E}_i)_{i=1}^{\check{m}}, \check{\mathscr{D}}_0 \colon (\check{z}_i : \check{E}_i)_{i=1}^{\check{m}}, \Delta \Rightarrow C$, where

$$\check{\mathscr{D}}_1^{\check{m}} = \mathscr{D}'^{n}_1, \mathscr{D}''^{p}_1,$$
$$(\check{z}_i : \check{E}_i)_{i=1}^{\check{m}} = (w_i : F_i)_{i=1}^n, (v_i : G_i)_{i=1}^p,$$

as described in Case 2.1 of the new method. Let $\mu(i)$ and $\mathscr{M}_i$ be as in the description of $\text{prune}(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. For $i = 1, \dots, n$, we have $\mu(i) = i$ and $\mathscr{D}_i = \mathscr{D}_i'$ is an $\varnothing$-interpolant to itself via $\mathscr{M}_i$. For $i = 1, \dots, p$, $\mathscr{D}_{\mu(n+i)}$ is an $\varnothing$-interpolant to $\mathscr{D}_i''$ via $\mathscr{M}_{n+i}$.

We define subsets $S_1^{\widetilde{m}}$ of $\{1,\dots,m\}$ and deductions $(\mathscr{E}_i\colon (z_j : E_j)_{j\in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ as follows:

$$S_i = \begin{cases} S_i' & \text{if } i \in M', \\ \{\,\mu(n+j) \mid j \in S_i''\,\} & \text{if } i \in M'' - M', \end{cases}$$

$$\mathscr{E}_i = \begin{cases} |\mathscr{E}_i'[(\mathscr{M}_j/w_j)_{j\in S_i'}]|_\beta & \text{if } i \in M', \\ |\mathscr{E}_i''[(\mathscr{M}_{n+j}/v_j)_{j\in S_i''}]|_\beta & \text{if } i \in M'' - M'. \end{cases}$$

We show that $S_1^{\widetilde{m}}$ and $\mathscr{E}_1^{\widetilde{m}}$ satisfy conditions 2–4.

Condition 2 follows from (28.ii) and (29.ii), using the property of $\mathscr{M}_i$ mentioned above. Condition 4 is a consequence of (23).

It remains to prove condition 3. Since $\mathscr{D}_0 = |\check{\mathscr{D}}_0[(\mathscr{M}_i/\check{z}_i)_{i=1}^{\check{m}}]|_\beta$, $\mathscr{D}_0$ is the normal form of

$$\dfrac{\begin{matrix}(z_i : E_i)_{i=1}^n, \Delta' \\ \mathscr{D}_0'[(\mathscr{M}_i/w_i)_{i=1}^n] \\ C'' \to C\end{matrix} \qquad \begin{matrix}(z_{\mu(n+i)} : E_{\mu(n+i)})_{i=1}^p, \Delta'' \\ \mathscr{D}_0''[(\mathscr{M}_{n+i}/v_i)_{i=1}^p] \\ C''\end{matrix}}{C} \to E$$

Since $\widetilde{\mathscr{D}}_0$ is of the form (27), it suffices to show

$$\widetilde{\mathscr{D}}_0'[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}] =_\beta \mathscr{D}_0'[(\mathscr{M}_i/w_i)_{i=1}^n], \tag{30}$$

$$\widetilde{\mathscr{D}}_0''[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}] =_\beta \mathscr{D}_0''[(\mathscr{M}_{n+i}/v_i)_{i=1}^p]. \tag{31}$$

We can show (30) as follows:

$$\begin{aligned}\widetilde{\mathscr{D}}_0'[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}] &=_\beta \widetilde{\mathscr{D}}_0'[(\mathscr{E}_i'[(\mathscr{M}_j/w_j)_{j\in S_i'}]/\widetilde{z}_i]_{i\in M'}] \\ &= \widetilde{\mathscr{D}}_0'[(\mathscr{E}_i'/\widetilde{z}_i)_{i\in M'}][(\mathscr{M}_i/w_i)_{i=1}^n] \quad \text{by (28.i)} \\ &=_\beta \mathscr{D}_0'[(\mathscr{M}_i/w_i)_{i=1}^n] \quad \text{by (28.iii)}.\end{aligned}$$

It remains to prove (31). Since $(\widetilde{\mathscr{D}}_i)_{i\in M''}$ is an interpolant to $\mathscr{D}''$ with respect to the partition $(\Gamma''; \Delta'')$ via $\widetilde{\mathscr{D}}_0''$, condition 2 implies that

(32) $(\mathscr{D}_j)_{j\in\bigcup_{i\in M''} S_i}$ is an interpolant to $\mathscr{D}''$ with respect to the partition $(\Gamma''; \Delta'')$ via the normal form of $\widetilde{\mathscr{D}}_0''[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}]\colon (z_j : E_j)_{j\in\bigcup_{i\in M''} S_i}, \Delta'' \Rightarrow C''$.

Applying the induction hypothesis again to $\mathscr{D}''$ with respect to (32) and noting Lemma 32, we obtain elements $(\tau(j))_{j\in\bigcup_{i\in M''} S_i}$ of $\{1,\dots,p\}$ and deductions $(\mathscr{T}_j\colon v_{\tau(j)} : G_{\tau(j)} \Rightarrow E_j)_{j\in\bigcup_{i\in M''} S_i}$ such that

(33) i. $\{\,\tau(j) \mid j \in \bigcup_{i\in M''} S_i\,\} = \{1,\dots,p\}$;

ii. $\mathscr{D}_{\tau(j)}''$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{T}_j$ for each $j \in \bigcup_{i\in M''} S_i$;

iii. $\widetilde{\mathscr{D}}_0''[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}][(\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}] \twoheadrightarrow_\beta \mathscr{D}_0''$.

By (33.ii) and part 2 of Claim B, for $j \in \bigcup_{i\in M''} S_i$, we have $\mu(n+\tau(j)) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{D}_{\tau(j)}''$ via $\mathscr{M}_{n+\tau(j)}\colon z_j : E_j \Rightarrow G_{\tau(j)}$. It follows that $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{\tau(j)}]\colon z_j : E_j \Rightarrow E_j$. Hence by condition 4,

$$\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{\tau(j)}]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M''. \tag{34}$$

Now

$$\begin{aligned}
\widetilde{\mathscr{D}}_0''[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}] &=_\beta \widetilde{\mathscr{D}}_0''[(\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{\tau(j)}]/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}] \quad \text{by (34)}\\
&= \widetilde{\mathscr{D}}_0''[(\mathscr{E}_i[(\mathscr{T}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}][(\mathscr{M}_{n+i}/v_i)_{i=1}^{p}] \quad \text{by (33.i)}\\
&= \widetilde{\mathscr{D}}_0''[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}][(\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}][(\mathscr{M}_{n+i}/v_i)_{i=1}^{p}]\\
&\twoheadrightarrow_\beta \mathscr{D}_0''[(\mathscr{M}_{n+i}/v_i)_{i=1}^{p}] \quad \text{by (33.iii)}
\end{aligned}$$

We have proved condition 3.

Case 2.2. The main branch of $\mathscr{D}'$ leads to an assumption belonging to Γ'. Then by Lemma 40, the main branch of $\widetilde{\mathscr{D}}_0$ leads to some $\widetilde{z}_i : \widetilde{E}_i$, say $\widetilde{z}_1 : \widetilde{E}_1$. Since $\widetilde{\mathscr{D}}_0$ cannot end in $\to I$, $\widetilde{\mathscr{D}}_0$ must have the following form:

$$\text{(35)} \qquad \widetilde{\mathscr{D}}_0 = \cfrac{\widetilde{z}_1 : \widetilde{C}_1^{\widetilde{k}} \to C \qquad \begin{pmatrix} (\widetilde{z}_j : \widetilde{E}_j)_{j\in M_i}, \widetilde{\Delta}_i \\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i \end{pmatrix}_{i=1}^{\widetilde{k}}}{C} \to E$$

where $\widetilde{C}_1^{\widetilde{k}} \to C = \widetilde{E}_1$ and

$$\{1\} \cup M_1 \cup \cdots \cup M_{\widetilde{k}} = \{1, \dots, \widetilde{m}\},$$
$$\widetilde{\Delta}_1 \cup \cdots \cup \widetilde{\Delta}_{\widetilde{k}} = \Delta.$$

Since $\widetilde{\mathscr{D}}_0$ satisfies condition (I4) of Definition 16, each $\widetilde{\mathscr{C}}_i$ must satisfy the following condition:

(D) Every maximal path in $\widetilde{\mathscr{C}}_i$ that starts inside the endformula $\widetilde{C}_i$ or some $\widetilde{z}_j : \widetilde{E}_j$ must end inside an assumption belonging to $\widetilde{\Delta}_i$.

$\widetilde{\mathscr{D}}_1$ must have the following form:

$$\text{(36)} \qquad \widetilde{\mathscr{D}}_1 = \cfrac{\begin{matrix}\widetilde{\Gamma}_1, (\widetilde{u}_i : \widetilde{C}_i)_{i=1}^{\widetilde{k}_1}\\ \widetilde{\mathscr{D}}_1^- \\ \widetilde{C}_{\widetilde{k}_1+1}^{\widetilde{k}} \to C\end{matrix}}{\widetilde{C}_1^{\widetilde{k}} \to C} \to I, \widetilde{u}_1^{\widetilde{k}_1}$$

where $0 \le \widetilde{k}_1 \le \widetilde{k}$ and $\widetilde{\mathscr{D}}_1^-$ does not end in $\to I$.

Case 2.2a. $\widetilde{k}_1 < \widetilde{k}$. Then Lemma 40 (part 3) implies that $\widetilde{C}_{\widetilde{k}} = C''$, and it is easy to see the following, using (D):

(37) a. $\widetilde{\Delta}_{\widetilde{k}} = \Delta''$;

b. $(\widetilde{\mathscr{D}}_j)_{j\in M_{\widetilde{k}}}$ is an interpolant to $\mathscr{D}''$ with respect to $(\Gamma''; \Delta'')$ via $\widetilde{\mathscr{C}}_{\widetilde{k}}$;

c. $\widetilde{\Delta}_1 \cup \cdots \cup \widetilde{\Delta}_{\widetilde{k}-1} = \Delta'$;

d. $(\widetilde{\mathscr{D}}_j)_{j\in\{1\}\cup\bigcup_{i=1}^{\widetilde{k}-1} M_i}$ is an interpolant to $\mathscr{D}'$ with respect to $(\Gamma'; \Delta')$ via

$$\widetilde{\mathscr{D}}_0' = \cfrac{\widetilde{z}_1 : \widetilde{C}_1^{\widetilde{k}-1} \to C'' \to C \qquad \begin{pmatrix} (\widetilde{z}_j : \widetilde{E}_j)_{j\in M_i}, \widetilde{\Delta}_i \\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i \end{pmatrix}_{i=1}^{\widetilde{k}-1}}{C'' \to C} \to E$$

By (D), the main branch of $\widetilde{\mathscr{C}_{\tilde{k}}}$ leads to an assumption belonging to Δ''. It follows that the main branch of $\mathscr{D}''$ leads to an assumption belonging to Δ'', i.e., Case 2.2.1 of the description of the new method applies.

Using (D) again, we can see that $\widetilde{\mathscr{C}_{\tilde{k}}}$ must be of the form

$$\widetilde{\mathscr{C}_{\tilde{k}}} = \qquad \begin{array}{c} (\widetilde{z}_i : \widetilde{E}_i)_{i \in M_{\tilde{k}}}, (u_j : A_j)_{j=1}^{l}, \Delta'' \\ \widetilde{\mathscr{B}}_0 \\ \dfrac{B}{A_1^l \to B} \to I, u_1^l \end{array}$$

where every maximal path in $\widetilde{\mathscr{B}}_0$ starting inside some $u_j : A_j$ ends inside an assumption belonging to Δ''. Therefore,

(38) $(\widetilde{\mathscr{D}}_j)_{j \in M_{\tilde{k}}}, (u_j{:}A_j)_{j=1}^{l}$ is an interpolant to $\mathscr{B}$ with respect to $(\Gamma'', (u_j{:}A_j)_{j=1}^{l}; \Delta'')$ via $\widetilde{\mathscr{B}}_0[(\widetilde{y}_j : A_j/u_j)_{j=1}^{l}] \colon (\widetilde{z}_i : \widetilde{E}_i)_{i \in M_{\tilde{k}}}, (\widetilde{y}_j : A_j)_{j=1}^{l}, \Delta'' \Rightarrow B.$

Let

$$M' = \{1\} \cup \bigcup_{i=1}^{\tilde{k}-1} M_i, \qquad M'' = M_{\tilde{k}}.$$

We apply the induction hypothesis to $\mathscr{D}'$ with respect to (37.d) and to $\mathscr{B}$ with respect to (38). It is easy to see that $\hat{l} = 0$, i.e., Case 2.2.1.1 of the description of the new method applies, and we obtain subsets $(S_i')_{i \in M'}$ of $\{1, \dots, n\}$, subsets $(S_i'')_{i \in M''}$ of P, and deductions $(\mathscr{E}_i' \colon (w_j : F_j)_{j \in S_i'} \Rightarrow \widetilde{E}_i)_{i \in M'}$, $(\mathscr{E}_i'' \colon (v_j : G_j)_{j \in S_i''} \Rightarrow \widetilde{E}_i)_{i \in M''}$ such that

(39) i. $\bigcup_{i \in M'} S_i' = \{1, \dots, n\}$;

ii. for each $i \in M'$, $(\mathscr{D}_j')_{j \in S_i'}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i'$;

iii. $\widetilde{\mathscr{D}}_0'[(\mathscr{E}_i'/\widetilde{z}_i)_{i \in M'}] \twoheadrightarrow_\beta \mathscr{D}_0'$;

iv. for each $i \in M'$ and for each $j \in S_i'$, $\mathscr{E}_i'$ is long for $\mathscr{D}_j'$ with respect to $w_j : F_j$;

(40) i. $\bigcup_{i \in M''} S_i'' = P$;

ii. for each $i \in M''$, $(\mathscr{B}_j)_{j \in S_i''}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i''$;

iii. $\widetilde{\mathscr{B}}_0[(\mathscr{E}_i''/\widetilde{z}_i)_{i \in M''}] \twoheadrightarrow_\beta \hat{\mathscr{B}}_0$;

iv. for each $i \in M''$ and for each $j \in S_i''$, $\mathscr{E}_i''$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$.

The output $\mathscr{D}_1^m, \mathscr{D}_0$ of the new method is the result of applying the pruning procedure to $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow \check{E}_i)_{i=1}^{\check{m}}, \check{\mathscr{D}}_0 \colon (\check{z}_i : \check{E}_i)_{i=1}^{\check{m}}, \Delta \Rightarrow C$, where

$$\check{\mathscr{D}}_1^{\check{m}} = \mathscr{D}'^{n}_1, (\mathscr{B}_i)_{i \in P},$$
$$(\check{z}_i : \check{E}_i)_{i=1}^{\check{m}} = (w_i : F_i)_{i=1}^{n}, (v_i : G_i)_{i \in P},$$

as described in Case 2.2.1.1 of the new method. Let $p_1, \dots, p_s$ list the elements of P in increasing order, so that $\check{m} = n + s$. Let $\mu(i)$ and $\mathscr{M}_i$ be as in the description of prune$(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. For $i = 1, \dots, n$, we have $\mu(i) = i$ and $\mathscr{D}_i = \mathscr{D}_i'$ is an $\varnothing$-interpolant to itself via $\mathscr{M}_i$. For $i = 1, \dots, s$, $\mathscr{D}_{\mu(n+i)}$ is an $\varnothing$-interpolant to $\mathscr{B}_{p_i}$ via $\mathscr{M}_{n+i}$.

We define subsets $S_1^{\widetilde{m}}$ of $\{1,\ldots,m\}$ and deductions $(\mathscr{E}_i\colon (z_j : E_j)_{j\in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ as follows:

$$S_i = \begin{cases} S_i' & \text{if } i \in M', \\ \{\,\mu(n+j) \mid p_j \in S_i''\,\} & \text{if } i \in M'' - M', \end{cases}$$

$$\mathscr{E}_i = \begin{cases} |\mathscr{E}_i'[(\mathscr{M}_j/w_j)_{j\in S_i'}]|_\beta & \text{if } i \in M', \\ |\mathscr{E}_i''[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in S_i''}]|_\beta & \text{if } i \in M'' - M'. \end{cases}$$

The proof of conditions 2–4 is entirely analogous to Case 2.1. We leave the details to the reader.

Case 2.2b. $\widetilde{k}_1 = \widetilde{k}$. In this case $\widetilde{\mathscr{D}}_1$ must look like

$$\widetilde{\mathscr{D}}_1 = \dfrac{\dfrac{\begin{matrix}\widetilde{\Gamma}_1', (\widetilde{u}_i : \widetilde{C}_i)_{i\in K'} \\ \widetilde{\mathscr{D}}_1' \\ (A_1^l \to B) \to C\end{matrix} \qquad \begin{matrix}\widetilde{\Gamma}_1'', (\widetilde{u}_i : \widetilde{C}_i)_{i\in K''} \\ \widetilde{\mathscr{D}}_1'' \\ A_1^l \to B\end{matrix}}{C}\ {\to}E}{\widetilde{C}_1^k \to C}\ {\to}I, \widetilde{u}_1^k \tag{41}$$

where

$$\widetilde{\Gamma}_1' \cup \widetilde{\Gamma}_1'' = \widetilde{\Gamma}_1,$$
$$K' \cup K'' = \{1,\ldots,\widetilde{k}\}.$$

By Lemma 40, the main branch of $\widetilde{\mathscr{D}}_1'$ leads to an assumption belonging to $\widetilde{\Gamma}_1'$. Since $\widetilde{\mathscr{D}}_1'$ does not end in $\to I$ and since $\widetilde{\mathscr{D}}_1$ satisfies condition (I3) of Definition 16, we have

(E) Every maximal path in $\widetilde{\mathscr{D}}_1'$ starting inside the endformula $(A_1^l \to B) \to C$ or some $\widetilde{u}_i : \widetilde{C}_i$ leads to an assumption belonging to $\widetilde{\Gamma}_1'$.

Since $\widetilde{\mathscr{D}}_0[(\widetilde{\mathscr{D}}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] \twoheadrightarrow_\beta \mathscr{D}$, we have

$$\widetilde{\mathscr{D}}_1'[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K'}] \twoheadrightarrow_\beta \mathscr{D}', \tag{42}$$

$$\widetilde{\mathscr{D}}_1''[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K''}] \twoheadrightarrow_\beta \mathscr{D}'', \tag{43}$$

which implies that

$$\bigcup_{i\in K'} \widetilde{\Delta}_i = \Delta', \quad \bigcup_{i\in K''} \widetilde{\Delta}_i = \Delta''.$$

Let

$$M' = \bigcup_{i\in K'} M_i, \quad M'' = \bigcup_{i\in K''} M_i. \tag{44}$$

Then we have

$$\{1\} \cup M' \cup M'' = \{1,\ldots,\widetilde{m}\},$$
$$\widetilde{\Gamma}_1' \cup \bigcup_{i\in M'} \widetilde{\Gamma}_i = \Gamma', \quad \widetilde{\Gamma}_1'' \cup \bigcup_{i\in M''} \widetilde{\Gamma}_i = \Gamma''.$$

Let

$$(45)\qquad \widetilde{\mathscr{A}} = \dfrac{\begin{array}{c}\widetilde{\Gamma}'_1, (\widetilde{u}_i : \widetilde{C}_i)_{i\in K'}\\ \widetilde{\mathscr{D}}'_1 \\ (A_1^l \to B)\to C\end{array}}{(\widetilde{C}_i)_{i\in K'} \to (A_1^l\to B)\to C}\to I, (\widetilde{u}_i)_{i\in K'}$$

$$(46)\qquad \widetilde{\mathscr{D}}'_0 = \dfrac{\widetilde{w}_1 : (\widetilde{C}_i)_{i\in K'} \to (A_1^l\to B)\to C \quad \left(\begin{array}{c}(\widetilde{z}_j:\widetilde{E}_j)_{j\in M_i}, \widetilde{\Delta}_i\\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i\end{array}\right)_{i\in K'}}{(A_1^l\to B)\to C}\to E$$

where $\widetilde{\mathscr{C}}_i$ are as in (35). We show

(47) $\widetilde{\mathscr{A}}, (\widetilde{\mathscr{D}}_i)_{i\in M'}$ is an interpolant to $\mathscr{D}'$ with respect to $(\Gamma';\Delta')$ via $\widetilde{\mathscr{D}}'_0$.

Firstly,

$$\begin{aligned}\widetilde{\mathscr{D}}'_0[\widetilde{\mathscr{A}}/\widetilde{w}_1, (\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M'}] &\twoheadrightarrow_\beta \widetilde{\mathscr{D}}'_1[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K'}]\\ &\twoheadrightarrow_\beta \mathscr{D}' \quad \text{by (42).}\end{aligned}$$

Secondly, $\widetilde{\mathscr{A}}$ satisfies condition (I3) of Definition 16 by (E). Finally, the property (D) ensures that $\widetilde{\mathscr{D}}'_0$ satisfies condition (I4) of Definition 16. So we have shown (47).

By the induction hypothesis, we have subsets $T, (S'_i)_{i\in M'}$ of $\{1,\dots,n\}$ and deductions $\mathscr{F} : (w_j : F_j)_{j\in T} \Rightarrow (\widetilde{C}_i)_{i\in K'} \to (A_1^l \to B)\to C$, $(\mathscr{E}'_i : (w_j : F_j)_{j\in S'_i} \Rightarrow \widetilde{E}_i)_{i\in M'}$ such that

(48) i. $T \cup \bigcup_{i\in M'} S'_i = \{1,\dots,n\}$;

- ii. a. $(\mathscr{D}'_j)_{j\in T}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{A}}$ via $\mathscr{F}$;
  - b. for each $i \in M'$, $(\mathscr{D}'_j)_{j\in S'_i}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}'_i$;
- iii. $\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}'_i/\widetilde{z}_i)_{i\in M'}] \twoheadrightarrow_\beta \mathscr{D}'_0$;
- iv. a. for each $j \in T$, $\mathscr{F}$ is long for $\mathscr{D}'_j$ with respect to $w_j : F_j$;
  - b. for each $i \in M'$ and for each $j \in S'_i$, $\mathscr{E}'_i$ is long for $\mathscr{D}'_j$ with respect to $w_j : F_j$.

By (48.ii.a), (48.iv.a), and part 2 of Lemma 44, $\widetilde{\mathscr{A}}$ and $\mathscr{F}$ have identical final blocks of applications of $\to I$. Since $\widetilde{\mathscr{A}}$ satisfies condition (I3) of Definition 16, it follows from (48.ii.a) and Lemma 21 that $\mathscr{F}$ also satisfies condition (I3). By (12), (46), and (48.iii), then, $\mathscr{F}$ must be of the following form:

$$(49)\qquad \mathscr{F} = \dfrac{\dfrac{w_1 : C_1^k \to (A_1^l\to B)\to C \quad \begin{array}{c}(w_j:F_j)_{j\in T^-}, (\widetilde{u}_j:\widetilde{C}_j)_{j\in K'}\\ \widehat{\mathscr{C}}_1^k\\ C_1^k\end{array}}{(A_1^l\to B)\to C}\to E}{(\widetilde{C}_j)_{j\in K'}\to(A_1^l\to B)\to C}\to I,(\widetilde{u}_j)_{j\in K'}$$

where

$$(50)\qquad \{1\} \cup T^- = T,$$

$$(51)\qquad T^- \cup \bigcup_{i\in M'} S'_i = N.$$

Let $\mathscr{F}^- : (w_i : F_i)_{i\in T}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \Rightarrow (A_1^l \to B) \to C$ be the following deduction:

$$(52)\qquad \mathscr{F}^- = \dfrac{w_1 : C_1^k \to (A_1^l \to B) \to C \qquad \begin{matrix}(w_j : F_j)_{j\in T^-}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \hat{\mathscr{C}}_1^k \\ C_1^k\end{matrix}}{(A_1^l \to B) \to C} \to E$$

Since $\mathscr{F}$ satisfies condition (I4) of Definition 16, each $\hat{\mathscr{C}_i}$ must satisfy the following condition:

(F) Every maximal path in $\hat{\mathscr{C}_i}$ that starts inside the endformula C_i or some $w_j : F_j$ must end inside some $\widetilde{u}_j : \widetilde{C}_j$.

Since $\mathscr{F}[(\mathscr{D}'_j/w_j)_{j\in T}] \twoheadrightarrow_\beta \widetilde{\mathscr{A}}$,

$$(53)\qquad \mathscr{F}^-[(\mathscr{D}'_j/w_j)_{j\in T}] = \dfrac{\begin{matrix}\Gamma'_1 \\ \mathscr{D}'_1 \\ C_1^k \to (A_1^l \to B) \to C\end{matrix} \qquad \begin{matrix}\bigcup_{j\in T^-}\Gamma'_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \hat{\mathscr{C}}_1^k[(\mathscr{D}'_j/w_j)_{j\in T^-}] \\ C_1^k\end{matrix}}{(A_1^l \to B) \to C} \to E \qquad \twoheadrightarrow_\beta \widetilde{\mathscr{D}}'_1$$

Also, by (48.iv.a), we have

(54) For $j \in T$, if $\mathscr{D}'_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I} : w_j : F_j \Rightarrow F_j$, then $\mathscr{F}^-[\mathscr{I}/w_j] \twoheadrightarrow_\beta \mathscr{F}^-$.

Case 2.2b.1. The main branch of $\mathscr{D}''$ leads to an assumption belonging to Δ'', i.e., Case 2.2.1 of the description of the new method applies. Then (43) implies that the main branch of $\widetilde{\mathscr{D}}''_1$ must lead to some $\widetilde{u}_{i_1} : \widetilde{C}_{i_1}$ ($i_1 \in K''$), and $\widetilde{\mathscr{D}}''_1$ must have the following form:

$$(55)\qquad \widetilde{\mathscr{D}}''_1 = \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}_1^{\widetilde{q}} \to A_{\widetilde{l}+1}^l \to B \qquad \left(\begin{matrix}\widetilde{\Gamma}''_{1,i}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}} \\ \widetilde{\mathscr{H}_i} \\ \widetilde{H}_i\end{matrix}\right)_{i=1}^{\widetilde{q}}}{A_{\widetilde{l}+1}^l \to B} \to E}{A_1^l \to B} \to I, u_1^{\widetilde{l}}$$

where

$$\widetilde{\Gamma}''_{1,1} \cup \cdots \cup \widetilde{\Gamma}''_{1,\widetilde{q}} = \widetilde{\Gamma}''_1,$$
$$\{i_1\} \cup K''_1 \cup \cdots \cup K''_{\widetilde{q}} = K''.$$

By (43), $\widetilde{\mathscr{C}_{i_1}}$ must have the following form:

$$(56)\qquad \widetilde{\mathscr{C}_{i_1}} = \dfrac{\dfrac{\begin{matrix}(\widetilde{z}_j : \widetilde{E}_j)_{j\in M_{i_1}}, \widetilde{\Delta}_{i_1}, (\widetilde{x}_i : \widetilde{H}_i)_{i=1}^{\widetilde{q}}, (u_j : A_j)_{j=\widetilde{l}+1}^{l} \\ \widetilde{\mathscr{C}}_{i_1}^- \\ B\end{matrix}}{A_{\widetilde{l}+1}^l \to B} \to I, u_{\widetilde{l}+1}^l}{\widetilde{H}_1^{\widetilde{q}} \to A_{\widetilde{l}+1}^l \to B} \to I, \widetilde{x}_1^{\widetilde{q}}$$

Since $\widetilde{\mathscr{D}}_1$ satisfies condition (I3) of Definition 16, each $\widetilde{\mathscr{H}_i}$ must satisfy the following condition:

(G) Every maximal path in $\widetilde{\mathscr{H}_i}$ that starts inside the endformula $\widetilde{H}_i$ or some $\widetilde{u}_j : \widetilde{C}_j$ must end inside an assumption belonging to $\widetilde{\Gamma}''_{1,i}$ or some $u_j : A_j$.

Let

$$(57)\quad \widetilde{\mathscr{D}}_1''^{-} = \dfrac{\widetilde{u}_{i_1} : \widetilde{H}_1^{\widetilde{q}} \to A^l_{\widetilde{l}+1} \to B \qquad \left(\begin{matrix} \widetilde{\Gamma}''_{1,i}, (\widetilde{u}_j : \widetilde{C}_j)_{j \in K''_i}, ((u_j : A_j)^\circ)^{\widetilde{l}}_{j=1} \\ \widetilde{\mathscr{H}_i} \\ \widetilde{H}_i \end{matrix}\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1} \to B} \to E$$

By (43), we get

$$(58)\quad \widetilde{\mathscr{D}}_1''^{-}[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j \in M_i}]/\widetilde{u}_i)_{i \in K''}] \twoheadrightarrow_\beta \begin{matrix} \Gamma'', \Delta'', ((u_j : A_j)^\circ)^{\hat{l}}_{j=1}, (u_j : A_j)^l_{j=\hat{l}+1} \\ \mathscr{B} \\ \dfrac{B}{A^l_{\widetilde{l}+1} \to B} \to I, u^l_{\widetilde{l}+1} \end{matrix}$$

where $\mathscr{B}$ is as in (7).

For $i = 1, \ldots, \widetilde{q}$, let

$$(59)\quad \widetilde{\mathscr{B}}_i = \begin{matrix} \widetilde{\Gamma}''_{1,i}, (\widetilde{u}_j : \widetilde{C}_j)_{j \in K''_i}, ((u_j : A_j)^\circ)^{\widetilde{l}}_{j=1} \\ \widetilde{\mathscr{H}_i} \\ \dfrac{\widetilde{H}_i}{(\widetilde{C}_j)_{j \in K''_i} \to \widetilde{H}_i} \to I, (\widetilde{u}_j)_{j \in K''_i} \end{matrix}$$

Let $\widetilde{\mathscr{B}}_0 : (\widetilde{v}_i : (\widetilde{C}_j)_{j \in K''_i} \to \widetilde{H}_i)^{\widetilde{q}}_{i=1}, (\widetilde{z}_i : \widetilde{E}_i)_{i \in M''}, (\widetilde{y}_j : A_j)^l_{j=\widetilde{l}+1}, \Delta'' \Rightarrow B$ be the following deduction:

$$(60)\quad \widetilde{\mathscr{B}}_0 = \dfrac{\dfrac{\begin{matrix} (\widetilde{z}_j : \widetilde{E}_j)_{j \in M_{i_1}}, \widetilde{\Delta}_{i_1} \\ \widetilde{\mathscr{C}}_{i_1} \\ \widetilde{H}_1^{\widetilde{q}} \to A^l_{\widetilde{l}+1} \to B \end{matrix} \qquad \left(\dfrac{\widetilde{v}_i : (\widetilde{C}_j)_{j \in K''_i} \to \widetilde{H}_i \qquad \left(\begin{matrix} (\widetilde{z}_h : \widetilde{E}_h)_{h \in M_j}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}}_j \\ \widetilde{C}_j \end{matrix}\right)_{j \in K''_i}}{\widetilde{H}_i} \to E \right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1} \to B} \to E \qquad (\widetilde{y}_j : A_j)^l_{j=\widetilde{l}+1}}{B} \to E$$

where $\widetilde{\mathscr{C}_j}$ are as in (35). Note that $\widetilde{\mathscr{B}}_0$ normalizes in at most $\widetilde{q} + l - \widetilde{l}$ non-erasing β-reduction steps (use (D)). We show

(61) $\widetilde{\mathscr{B}}_1^{\widetilde{q}}, (\widetilde{\mathscr{D}}_i)_{i \in M''}, (u_j : A_j)^l_{j=\widetilde{l}+1}$ is an interpolant to $\mathscr{B}$ with respect to the partition $(\Gamma'', ((u_j : A_j)^\circ)^l_{j=1}; \Delta'')$ via $|\widetilde{\mathscr{B}}_0|_\beta$.

Firstly,

$$\widetilde{\mathscr{B}}_0[(\widetilde{\mathscr{B}}_i/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\widetilde{\mathscr{D}}_i/\widetilde{z}_i)_{i \in M''}, (u_j : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}]$$

$\twoheadrightarrow_\beta$

$$
\dfrac{\dfrac{\begin{array}{c}\bigcup_{j\in M_{i_1}}\widetilde{\Gamma}_j,\widetilde{\Delta}_{i_1}\\ \widetilde{\mathscr{C}}_{i_1}[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_{i_1}}]\\ \widetilde{H}_1^{\widetilde{q}}\to A^l_{\widetilde{l}+1}\to B\end{array}\quad \left(\dfrac{\begin{array}{c}\widetilde{\Gamma}''_{1,i},((u_j:A_j)^\circ)^{\widetilde{l}}_{j=1}\\ \widetilde{\mathscr{B}}_i\\ (\widetilde{C}_j)_{j\in K''_i}\to\widetilde{H}_i\end{array}\quad \left(\begin{array}{c}\bigcup_{h\in M_j}\widetilde{\Gamma}_h,\widetilde{\Delta}_j\\ \widetilde{\mathscr{C}}_j[(\widetilde{\mathscr{D}}_h/\widetilde{z}_h)_{h\in M_j}]\\ \widetilde{C}_j\end{array}\right)_{j\in K''_i}}{\widetilde{H}_i}\to E\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1}\to B}\to E\quad (u_j:A_j)^l_{j=\widetilde{l}+1}}{B}\to E
$$

$\twoheadrightarrow_\beta$ (by (59))

$$
\dfrac{\dfrac{\begin{array}{c}\bigcup_{j\in M_{i_1}}\widetilde{\Gamma}_j,\widetilde{\Delta}_{i_1}\\ \widetilde{\mathscr{C}}_{i_1}[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_{i_1}}]\\ \widetilde{H}_1^{\widetilde{q}}\to A^l_{\widetilde{l}+1}\to B\end{array}\quad \left(\begin{array}{c}\widetilde{\Gamma}''_{1,i},\bigcup_{j\in K''_i}\bigcup_{h\in M_j}\widetilde{\Gamma}_h,\bigcup_{j\in K''_i}\widetilde{\Delta}_j,((u_j:A_j)^\circ)^{\widetilde{l}}_{j=1}\\ \widetilde{\mathscr{H}}_i[(\widetilde{\mathscr{C}}_j[(\widetilde{\mathscr{D}}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K''_i}]\\ \widetilde{H}_i\end{array}\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1}\to B}\to E\quad (u_j:A_j)^l_{j=\widetilde{l}+1}}{B}\to E
$$

= (by (57))

$$
\dfrac{\begin{array}{c}\Gamma'',\Delta'',((u_j:A_j)^\circ)^{\hat{l}}_{j=1}\\ \widetilde{\mathscr{D}}_1''^{-}[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K''}]\\ A^l_{\widetilde{l}+1}\to B\end{array}\quad (u_j:A_j)^l_{j=\widetilde{l}+1}}{B}\to E
$$

$\twoheadrightarrow_\beta$ (by (58))

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma'',\Delta'',((u_j:A_j)^\circ)^{\hat{l}}_{j=1},(u_j:A_j)^l_{j=\hat{l}+1}\\ \mathscr{B}\\ B\end{array}}{A^l_{\widetilde{l}+1}\to B}\to I,u^l_{\widetilde{l}+1}\quad (u_j:A_j)^l_{j=\widetilde{l}+1}}{B}\to E
$$

$\twoheadrightarrow_\beta$

$$\mathscr{B}.$$

Secondly, the property (G) ensures that each $\widetilde{\mathscr{B}}_i$ satisfies condition (I3) of Definition 16. Finally, the property (D) ensures that $\widetilde{\mathscr{B}}_0$ satisfies condition (I4) of Definition 16. So we have shown (61).

Case 2.2b.1.1. $\hat{l} = 0$, i.e., Case 2.2.1.1 of the description of the new method applies. Applying the induction hypothesis to $\mathscr{B}$ with respect to (61), we obtain subsets $V_1^{\widetilde{q}}, (S''_i)_{i\in M''}$ of P, deductions $(\mathscr{G}_i\colon (v_j{:}G_j)_{j\in V_i}, ((v_{a_j}{:}A_j)^\circ)^{\widetilde{l}}_{j=1} \Rightarrow (\widetilde{C}_j)_{j\in K''_i}\to \widetilde{H}_i)^{\widetilde{q}}_{i=1}$, and deductions $(\mathscr{E}''_i\colon (v_j:G_j)_{j\in S''_i} \Rightarrow \widetilde{E}_i)_{i\in M''}$ such that

(62) i. $\bigcup_{i=1}^{\widetilde{q}} V_i \cup \bigcup_{i\in M''} S''_i = P$;

ii. a. for each $i = 1,\dots,\widetilde{q}$, $(\mathscr{B}_j)_{j\in V_i}, ((u_j:A_j)^\circ)^{\widetilde{l}}_{j=1}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{B}}_i$ via $\mathscr{G}_i$;

b. for each $i \in M''$, $(\mathscr{B}_j)_{j\in S''_i}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}''_i$;

iii. $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}''_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j}/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}] \twoheadrightarrow_\beta \mathscr{B}_0$;

iv. a. for $i = 1, \ldots, \widetilde{q}$ and for $j \in V_i$, $\mathscr{G}_i$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$;

b. for $i \in M''$ and for $j \in S''_i$, $\mathscr{E}''_i$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$.

Let $\widetilde{\mathscr{G}_i} = \mathscr{G}_i[(u_j : A_j/v_{a_j})_{j=1}^{\widetilde{l}}]$. Since $\mathscr{G}_i$ satisfies condition (I4) of Definition 16, we have

(H) Every maximal path in $\widetilde{\mathscr{G}_i}$ that starts inside some $v_j : G_j$ ($j \in V_i$) or $u_j : A_j$ ($1 \le j \le \widetilde{l}$) must end inside the endformula $(\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i$.

Note that since $\widetilde{\mathscr{B}}_i$ satisfies condition (I3) of Definition 16, it follows from (62.ii.a) and Lemma 21 that $\mathscr{G}_i$ also satisfies condition (I3).

Let $V = \bigcup_{i=1}^{\widetilde{q}} V_i$. Let $\hat{\mathscr{E}}_1 \colon (w_j : F_j)_{j\in T}, (v_j : G_j)_{j\in V} \Rightarrow \widetilde{C}_1^{\widetilde{k}} \to C$ be the following deduction:

(63) $\hat{\mathscr{E}}_1 =$

$$\dfrac{\dfrac{\begin{matrix}(w_j : F_j)_{j\in T}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'}\\ \mathscr{F}^-\\ (A_1^l \to B) \to C\end{matrix} \qquad \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}_1^{\widetilde{q}} \to A^l_{\widetilde{l}+1} \to B \qquad \left(\dfrac{\begin{matrix}(v_j : G_j)_{j\in V_i}, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}}\\ \widetilde{\mathscr{G}_i}\\ (\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i \quad (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}\end{matrix}}{\widetilde{H}_i}\to E\right)_{i=1}^{\widetilde{q}}}{A^l_{\widetilde{l}+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\widetilde{l}}}{C}\to E}{\widetilde{C}_1^{\widetilde{k}} \to C}\to I, \widetilde{u}_1^{\widetilde{k}}$$

where $\mathscr{F}^-$ is as in (52). Since $\widetilde{\mathscr{G}_i}$ satisfies condition (I3), $\hat{\mathscr{E}}_1$ normalizes by a sequence of non-erasing β-reduction steps. We can show

(64) $(\mathscr{D}'_j)_{j\in T}, (\mathscr{B}_j)_{j\in V}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_1$ via $|\hat{\mathscr{E}}_1|_\beta$.

That $\hat{\mathscr{E}}_1$ satisfies condition (I4) of Definition 16 can be checked using (F) and (H). It remains to show $\hat{\mathscr{E}}_1[(\mathscr{D}'_j/w_j)_{j\in T}, (\mathscr{B}_j/v_j)_{j\in V}] \twoheadrightarrow_\beta \widetilde{\mathscr{D}}_1$.

$$\hat{\mathscr{E}}_1[(\mathscr{D}'_j/w_j)_{j\in T}, (\mathscr{B}_j/v_j)_{j\in V}]$$

$=$

$$\dfrac{\dfrac{\begin{matrix}\bigcup_{j\in T}\Gamma'_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'}\\ \mathscr{F}^-[(\mathscr{D}'_j/w_j)_{j\in T}]\\ (A_1^l \to B) \to C\end{matrix} \qquad \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}_1^{\widetilde{q}} \to A^l_{\widetilde{l}+1} \to B \qquad \left(\dfrac{\begin{matrix}\bigcup_{j\in V_i}\Gamma''_j, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}}\\ \widetilde{\mathscr{G}_i}[(\mathscr{B}_j/v_j)_{j\in V_i}]\\ (\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i \quad (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}\end{matrix}}{\widetilde{H}_i}\to E\right)_{i=1}^{\widetilde{q}}}{A^l_{\widetilde{l}+1} \to B}\to E}{A_1^l \to B}\to I, u_1^{\widetilde{l}}}{C}\to E}{\widetilde{C}_1^{\widetilde{k}} \to C}\to I, \widetilde{u}_1^{\widetilde{k}}$$

$\twoheadrightarrow_\beta$ (by (53))

$$\dfrac{\begin{matrix}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A^l_1 \to B) \to C\end{matrix} \qquad \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}^{\widetilde{q}}_1 \to A^l_{\widetilde{l}+1} \to B \quad \left(\dfrac{\begin{matrix}\bigcup_{j\in V_i} \Gamma''_j, ((u_j : A_j)^\circ)^{\widetilde{l}}_{j=1} \\ \widetilde{\mathscr{G}}_i[(\mathscr{B}_j/v_j)_{j\in V_i}] \\ (\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i\end{matrix} \quad (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}}{\widetilde{H}_i} \to E\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1} \to B} \to E}{A^l_1 \to B} \to I, u^{\widetilde{l}}_1}{\dfrac{C}{\widetilde{C}^k_1 \to C} \to I, \widetilde{u}^k_1} \to E$$

$\twoheadrightarrow_\beta$ (by (62.ii.a))

$$\dfrac{\begin{matrix}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A^l_1 \to B) \to C\end{matrix} \qquad \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}^{\widetilde{q}}_1 \to A^l_{\widetilde{l}+1} \to B \quad \left(\dfrac{\begin{matrix}\widetilde{\Gamma}''_{1,i}, ((u_j : A_j)^\circ)^{\widetilde{l}}_{j=1} \\ \widetilde{\mathscr{B}}_i \\ (\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i\end{matrix} \quad (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}}{\widetilde{H}_i} \to E\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1} \to B} \to E}{A^l_1 \to B} \to I, u^{\widetilde{l}}_1}{\dfrac{C}{\widetilde{C}^k_1 \to C} \to I, \widetilde{u}^k_1} \to E$$

$\twoheadrightarrow_\beta$ (by (59))

$$\dfrac{\begin{matrix}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A^l_1 \to B) \to C\end{matrix} \qquad \dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}^{\widetilde{q}}_1 \to A^l_{\widetilde{l}+1} \to B \quad \left(\begin{matrix}\widetilde{\Gamma}''_{1,i}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}, ((u_j : A_j)^\circ)^{\widetilde{l}}_{j=1} \\ \widetilde{\mathscr{H}}_i \\ \widetilde{H}_i\end{matrix}\right)^{\widetilde{q}}_{i=1}}{A^l_{\widetilde{l}+1} \to B} \to E}{A^l_1 \to B} \to I, u^{\widetilde{l}}_1}{\dfrac{C}{\widetilde{C}^k_1 \to C} \to I, \widetilde{u}^k_1} \to E$$

= (by (41) and (55))

$$\widetilde{\mathscr{D}}_1$$

The output $\mathscr{D}^m_1, \mathscr{D}_0$ of the new method is the result of applying the pruning procedure to $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow \check{E}_i)^{\check{m}}_{i=1}, \check{\mathscr{D}}_0 \colon (\check{z}_i : \check{E}_i)^{\check{m}}_{i=1}, \Delta \Rightarrow C$, where

$$\check{\mathscr{D}}^{\check{m}}_1 = \mathscr{D}'^n_1, (\mathscr{B}_i)_{i\in P},$$
$$(\check{z}_i : \check{E}_i)^{\check{m}}_{i=1} = (w_i : F_i)^n_{i=1}, (v_i : G_i)_{i\in P},$$

as described in Case 2.2.1.1 of the new method. Let $p_1, \ldots, p_s$ list the elements of P in increasing order, so that $\check{m} = n + s$. Let $\mu(i)$ and $\mathscr{M}_i$ be as in the description of prune$(\check{\mathscr{D}}^{\check{m}}_1, \check{\mathscr{D}}_0)$. For $i = 1, \ldots, n$, $\mu(i) = i$ and $\mathscr{D}_i = \mathscr{D}'_i$ is an $\varnothing$-interpolant to itself via $\mathscr{M}_i$. For $i = 1, \ldots, s$, $\mathscr{D}_{\mu(n+i)}$ is an $\varnothing$-interpolant to $\mathscr{B}_{p_i}$ via $\mathscr{M}_{n+i}$. We define subsets $S^{\widetilde{m}}_1$ of $\{1, \ldots, m\}$ and deductions $\mathscr{E}_i \colon (z_j : E_j)_{j\in S_i} \Rightarrow \widetilde{E}_i)^{\widetilde{m}}_{i=1}$ as follows:

$$S_i = \begin{cases} T \cup \{\, \mu(n+j) \mid p_j \in V \,\} & \text{if } i = 1, \\ S'_i & \text{if } i \in M' - \{1\}, \\ \{\, \mu(n+j) \mid p_j \in S''_i \,\} & \text{if } i \in M'' - M' - \{1\}. \end{cases}$$

$$\mathscr{E}_i = \begin{cases} |\hat{\mathscr{E}}_1[(\mathscr{M}_j/w_j)_{j\in T}, (\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V}]|_\beta & \text{if } i = 1,\\ |\mathscr{E}_i'[(\mathscr{M}_j/w_j)_{j\in S_i'}]|_\beta & \text{if } i \in M' - \{1\},\\ |\mathscr{E}_i''[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in S_i''}]|_\beta & \text{if } i \in M'' - M' - \{1\}.\end{cases}$$

($\hat{\mathscr{E}}_1, \mathscr{E}_i', \mathscr{E}_i''$ are given in (63), (48), and (62), respectively.) We show that $S_1^{\widetilde{m}}$ and $\mathscr{E}_1^{\widetilde{m}}$ satisfy conditions 2–4.

Condition 2 follows from (64), (48.ii.b), and (62.ii.b), using the property of $\mathscr{M}_i$ mentioned above. Condition 4 easily follows from (23).

It remains to prove condition 3. From (35) and (63), we see that

$$\begin{aligned}\widetilde{\mathscr{D}}_0[(\mathscr{E}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] &=_\beta \widetilde{\mathscr{D}}_0[\hat{\mathscr{E}}_1[(\mathscr{M}_j/w_j)_{j\in T}, (\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V}]/\widetilde{z}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i=2}^{\widetilde{m}}] \\ &\twoheadrightarrow_\beta \frac{\widetilde{\mathscr{L}} \quad \widetilde{\mathscr{R}}}{C} \to E\end{aligned}$$

where

$$\widetilde{\mathscr{L}} = \begin{array}{c}(z_j : E_j)_{j\in T\cup\bigcup_{i\in M'} S_i}, \Delta' \\ \mathscr{F}^-[(\mathscr{M}_j/w_j)_{j\in T}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K'}] \\ (A_1^l \to B) \to C\end{array}$$

$$\widetilde{\mathscr{R}} = \cfrac{\cfrac{\begin{array}{c}(z_h : E_h)_{h\in\bigcup_{j\in M_{i_1}} S_j}, \widetilde{\Delta}_{i_1} \\ \widetilde{\mathscr{C}}_{i_1}[(\mathscr{E}_j/\widetilde{z}_j)_{j\in M_{i_1}}] \\ \widetilde{H}_1^{\widetilde{q}} \to A_{\widetilde{l}+1}^l \to B\end{array} \quad \left(\cfrac{\begin{array}{c}(z_{\mu(n+j)} : E_{\mu(n+j)})_{p_j\in V_i}, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}} \\ \widetilde{\mathscr{G}}_i[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V_i}] \\ (\widetilde{C}_j)_{j\in K_i''} \to \widetilde{H}_i\end{array} \quad \left(\begin{array}{c}(z_g : E_g)_{g\in\bigcup_{h\in M_j} S_h}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}] \\ \widetilde{C}_j\end{array}\right)_{j\in K_i''}}{\widetilde{H}_i} \to E\right)_{i=1}^{\widetilde{q}}}{A_{\widetilde{l}+1}^l \to B} \to E}{A_1^l \to B} \to I, u_1^{\widetilde{l}}$$

Since $\mathscr{D}_0 = |\check{\mathscr{D}}_0[(\mathscr{M}_i/\check{z}_i)_{i=1}^{\check{m}}]|_\beta$, where $\check{\mathscr{D}}_0$ is given in (9), $\mathscr{D}_0$ is the normal form of

$$\cfrac{\begin{array}{c}(w_i : F_i)_{i=1}^n, \Delta' \\ \mathscr{D}_0'[(\mathscr{M}_i/w_i)_{i=1}^n] \\ (A_1^l \to B) \to C\end{array} \qquad \cfrac{\begin{array}{c}(z_{\mu(n+i)} : E_{\mu(n+i)})_{p_i\in P}, (u_j : A_j)_{j=1}^l, \Delta'' \\ \hat{\mathscr{B}}_0[(\mathscr{M}_{n+i}/v_{p_i})_{p_i\in P}] \\ B\end{array}}{A_1^l \to B} \to I, u_1^l}{C} \to E$$

By (56) and (60),

$$\begin{aligned}\widetilde{\mathscr{R}} &=_\beta \cfrac{\cfrac{\cfrac{\begin{array}{c}(z_j : E_j)_{j\in\bigcup_{i\in M_{i_1}} S_i}, \widetilde{\Delta}_{i_1} \\ \widetilde{\mathscr{C}}_{i_1}[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_{i_1}}] \\ \widetilde{H}_1^{\widetilde{q}} \to A_{\widetilde{l}+1}^l \to B\end{array} \quad \left(\cfrac{\begin{array}{c}(z_{\mu(n+j)} : E_{\mu(n+j)})_{p_j\in V_i}, ((u_j : A_j)^\circ)_{j=1}^{\widetilde{l}} \\ \widetilde{\mathscr{G}}_i[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V_i}] \\ (\widetilde{C}_j)_{j\in K_i''} \to \widetilde{H}_i\end{array} \quad \left(\begin{array}{c}(z_g : E_g)_{g\in\bigcup_{h\in M_j} S_h}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}] \\ \widetilde{C}_j\end{array}\right)_{j\in K_i''}}{\widetilde{H}_i} \to E\right)_{i=1}^{\widetilde{q}}}{A_{\widetilde{l}+1}^l \to B} \to E \qquad (u_j : A_j)_{\widetilde{l}+1}^l}{B} \to E}{A_1^l \to B} \to I, u_1^l \\ &=_\beta \cfrac{\begin{array}{c}(z_h : E_h)_{h\in\{\mu(n+j)\mid p_j\in V\}\cup\bigcup_{i\in M''} S_i}, (u_j : A_j)_{j=1}^l, \Delta'' \\ \widetilde{\mathscr{B}}_0[(\widetilde{\mathscr{G}}_i[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (u_j : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^l] \\ B\end{array}}{A_1^l \to B} \to I, u_1^l\end{aligned}$$

So it suffices to show

(65) $$\mathscr{F}^{-}[(\mathscr{M}_j/w_j)_{j\in T}, (\widetilde{\mathscr{C}_j}[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K'}] =_\beta \mathscr{D}'_0[(\mathscr{M}_i/w_i)_{i=1}^n]$$

and

(66)
$$\widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}] =_\beta \mathscr{B}_0[(\mathscr{M}_{n+i}/v_{p_i})_{p_i\in P}].$$

We first show (65). By (47), (48.ii.a), and condition 2,

(67) $(\mathscr{D}'_j)_{j\in T}, (\mathscr{D}_j)_{j\in\bigcup_{i\in M'} S_i}$ is an interpolant to $\mathscr{D}'$ with respect to $(\Gamma';\Delta')$ via the normal form of $\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}]$: $(w_j : F_j)_{j\in T}, (z_j : E_j)_{j\in\bigcup_{i\in M'} S_i}, \Delta' \Rightarrow (A_1^l \to B) \to C$.

Note that

(68) $$\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}] \twoheadrightarrow_\beta \mathscr{F}^{-}[(\widetilde{\mathscr{C}_j}[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K'}].$$

Applying the induction hypothesis again to (67) and noting Lemma 32, we obtain elements $(\rho(j))_{j\in\bigcup_{i\in M'} S_i}$ of $\{1,\ldots,n\}$ and deductions $(\mathscr{I}_j : w_j : F_j \Rightarrow F_j)_{j\in T}$, $(\mathscr{P}_j\colon w_{\rho(j)} : F_{\rho(j)} \Rightarrow E_j)_{j\in\bigcup_{i\in M'} S_i}$ such that

(69) i. $T \cup \{\,\rho(j) \mid j \in \bigcup_{i\in M'} S_i\,\} = \{1,\ldots,n\}$;
ii. a. $\mathscr{D}'_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I}_j$ for each $j \in T$;
b. $\mathscr{D}'_{\rho(j)}$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{P}_j$ for each $j \in \bigcup_{i\in M'} S_i$;
iii. $\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}][(\mathscr{I}_j/w_j)_{j\in T}, (\mathscr{P}_j/z_j)_{j\in\bigcup_{i\in M'} S_i}] \twoheadrightarrow_\beta \mathscr{D}'_0$.

By (25), (69.ii.b) implies that for $j \in \bigcup_{i\in M'} S_i$, $\mu(\rho(j)) = \rho(j) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{P}_j[\mathscr{M}_j/w_j]$: $z_j : E_j \Rightarrow E_j$. Thus,

(70) $$T \cup \bigcup_{i\in M'} S_i = \{1,\ldots,n\},$$

and

(71) $$\mathscr{E}_i[(\mathscr{P}_j[\mathscr{M}_j/w_j]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M'.$$

Also, by (69.ii.a) and (48.iv.a),

(72) $$\mathscr{F}[(\mathscr{I}_j/w_j)_{j\in T}] \twoheadrightarrow_\beta \mathscr{F}.$$

Now we can show (65) as follows:

$$\begin{aligned}
&\mathscr{F}^{-}[(\mathscr{M}_j/w_j)_{j\in T}, (\widetilde{\mathscr{C}_j}[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K'}] \\
&=_\beta \widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}][(\mathscr{M}_j/w_j)_{j\in T}] \quad \text{by (68)} \\
&=_\beta \widetilde{\mathscr{D}}'_0[\mathscr{F}[(\mathscr{I}_j/w_j)_{j\in T}]/\widetilde{w}_1, (\mathscr{E}_i[(\mathscr{P}_j[\mathscr{M}_j/w_j]/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M'}][(\mathscr{M}_j/w_j)_{j\in T}] \quad \text{by (71) and (72)} \\
&= \widetilde{\mathscr{D}}'_0[\mathscr{F}[(\mathscr{I}_j/w_j)_{j\in T}]/\widetilde{w}_1, (\mathscr{E}_i[(\mathscr{P}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M'}][(\mathscr{M}_j/w_j)_{j=1}^n] \quad \text{by (70)} \\
&= \widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}][(\mathscr{I}_j/w_j)_{j\in T}, (\mathscr{P}_j/z_j)_{j\in\bigcup_{i\in M'} S_i}][(\mathscr{M}_j/w_j)_{j=1}^n] \\
&\twoheadrightarrow_\beta \mathscr{D}'_0[(\mathscr{M}_j/w_j)_{j=1}^n] \quad \text{by (69.iii).}
\end{aligned}$$

We now turn to (66). By (61), (62.ii.a) and condition 2,

(73) $(\mathscr{B}_j)_{j\in V}, (\mathscr{D}_j)_{j\in\bigcup_{i\in M''} S_i}, (u_j : A_j)_{j=1}^{l}$ is an interpolant to $\mathscr{B}$ with respect to $(\Gamma'', (u_j : A_j)_{j=1}^{l}; \Delta'')$ via the normal form of $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}]\colon (v_j : G_j)_{j\in V}, (z_j : E_j)_{j\in\bigcup_{i\in M''} S_i}, (v_{a_j} : A_j)_{j=1}^{l}, \Delta'' \Rightarrow B$.

Applying the induction hypothesis again to $\mathscr{B}$ with respect to (73) and noting Lemma 32, we obtain elements $(\tau(j))_{j\in\bigcup_{i\in M''} S_i}$ of $\{1, \ldots, s\}$ and deductions $(\mathscr{I}_j\colon v_j : G_j \Rightarrow G_j)_{j\in V}$, $(\mathscr{T}_j\colon v_{p_{\tau(j)}} : G_{p_{\tau(j)}} \Rightarrow E_j)_{j\in\bigcup_{i\in M''} S_i}$ such that

(74) i. $V \cup \{\, p_{\tau(j)} \mid j \in \bigcup_{i\in M''} S_i \,\} = P$;

ii. a. $\mathscr{B}_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I}_j$ for each $j \in V$;

b. $\mathscr{B}_{p_{\tau(j)}}$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{T}_j$ for $j \in \bigcup_{i\in M''} S_i$;

iii. $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j}{:}A_j)_{j=\widetilde{l}+1}^{l}][(\mathscr{I}_j/v_j)_{j\in V}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}] \twoheadrightarrow_\beta \mathscr{B}_0$.

By (25), (74.ii.b) implies that for $j \in \bigcup_{i\in M''} S_i$, $\mu(n + \tau(j)) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{B}_{p_{\tau(j)}}$ via $\mathscr{M}_{n+\tau(j)}\colon z_j : E_j \Rightarrow G_{p_{\tau(j)}}$. It follows that $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{p_{\tau(j)}}]\colon z_j : E_j \Rightarrow E_j$. Hence by condition 4,

$$\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{p_{\tau(j)}}]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M''. \tag{75}$$

Also, by (74.ii.a) and (62.iv.a),

$$\mathscr{G}_i[(\mathscr{I}_j/v_j)_{j\in V_i}] \twoheadrightarrow_\beta \mathscr{G}_i \quad \text{for } i = 1, \ldots, \widetilde{q}. \tag{76}$$

Now we can show (66) as follows:

$$\begin{aligned}
&\widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}]\\
&= \widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}][(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V}]\\
&=_\beta \widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{I}_j/v_j)_{j\in V_i}]/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{n+\tau(j)}/v_{p_{\tau(j)}}]/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}]\\
&\qquad [(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in V}] \quad \text{by (75) and (76)}\\
&= \widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{I}_j/v_j)_{j\in V_i}]/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i[(\mathscr{T}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}][(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in P}]\\
&\qquad\qquad \text{by (74.i)}\\
&= \widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}][(\mathscr{I}_j/v_j)_{j\in V}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}][(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in P}]\\
&\twoheadrightarrow_\beta \mathscr{B}_0[(\mathscr{M}_{n+j}/v_{p_j})_{p_j\in P}] \quad \text{by (74.iii)}.
\end{aligned}$$

Case 2.2b.1.2. $\hat{l} \geq 1$, i.e., Case 2.2.1.2 of the description of the new method applies. Applying the induction hypothesis to $\mathscr{B}$ with respect to (61), we conclude

$$\hat{l} \leq \widetilde{l},$$

and obtain subsets $U_1^{\widetilde{q}}$ of P^+, subsets $V_1^{\widetilde{q}}, (S_i'')_{i\in M''}$ of P^-, deductions $(\mathscr{G}_i\colon (v_j : G_j)_{j\in U_i\cup V_i}, ((v_{a_j} : A_j)^\circ)_{j=\hat{l}+1}^{\widetilde{l}} \Rightarrow (\widetilde{C}_j)_{j\in K_i''} \to \widetilde{H}_i)_{i=1}^{\widetilde{q}}$, and deductions $(\mathscr{E}_i''\colon (v_j : G_j)_{j\in S_i''} \Rightarrow \widetilde{E}_i)_{i\in M''}$ such that

(77) i. a. $\bigcup_{i=1}^{\widetilde{q}} U_i = P^+$;

b. $\bigcup_{i=1}^{\widetilde{q}} V_i \cup \bigcup_{i\in M''} S_i'' = P^-$;

ii. a. for each $i = 1, \dots, \widetilde{q}$, $(\mathscr{B}_j)_{j\in U_i\cup V_i}, ((u_j : A_j)^\circ)_{j=\hat{l}+1}^{\widetilde{l}}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{B}_i}$ via $\mathscr{G}_i$;

b. for each $i \in M''$, $(\mathscr{B}_j)_{j\in S_i''}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}_i}$ via $\mathscr{E}_i''$;

iii. $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)_{i=1}^{\widetilde{q}}, (\mathscr{E}_i''/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)_{j=\widetilde{l}+1}^{l}] \twoheadrightarrow_\beta \mathscr{B}_0$;

iv. a. for $i = 1, \dots, \widetilde{q}$ and for $j \in U_i \cup V_i$, $\mathscr{G}_i$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$;

b. for $i \in M''$ and for $j \in S_i''$, $\mathscr{E}_i''$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$.

Let $\widetilde{\mathscr{G}_i} = \mathscr{G}_i[(u_j : A_j/v_{a_j})_{j=\hat{l}+1}^{\widetilde{l}}]$. Since $\mathscr{G}_i$ satisfies condition (I4) of Definition 16, we have

(I) Every maximal path in $\widetilde{\mathscr{G}_i}$ that starts inside some $v_j : G_j$ ($j \in U_i \cup V_i$) or $u_j : A_j$ ($\hat{l} + 1 \leq j \leq \widetilde{l}$) must end inside the endformula $(\widetilde{C}_j)_{j\in K_i''} \to \widetilde{H}_i$.

Note that since $\widetilde{\mathscr{B}_i}$ satisfies condition (I3) of Definition 16, it follows from (77.ii.a) and Lemma 21 that $\mathscr{G}_i$ also satisfies condition (I3).

Let $V = \bigcup_{i=1}^{\widetilde{q}} V_i$. Let $\hat{\mathscr{E}}_1 : \check{z}_1 : C_1^k \to ((G_i)_{i\in P^+} \to A_{\hat{l}+1}^l \to B) \to C, (w_j : F_j)_{j\in T^-}, (v_j : G_j)_{j\in V} \Rightarrow \widetilde{C}_1^{\widetilde{k}} \to C$ be the following deduction:

(78) $\hat{\mathscr{E}}_1 =$

$$
\dfrac{
\dfrac{\check{z}_1 : C_1^k \to ((G_i)_{i\in P^+} \to A_{\hat{l}+1}^l \to B) \to C \qquad \begin{array}{c}(w_j : F_j)_{j\in T^-}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \hat{\mathscr{C}}_1^k \\ C_1^k\end{array}}{((G_i)_{i\in P^+} \to A_{\hat{l}+1}^l \to B) \to C}\to E
\qquad
\dfrac{\dfrac{\dfrac{\widetilde{u}_{i_1} : \widetilde{H}_1^{\widetilde{q}} \to A_{\widetilde{l}+1}^l \to B \qquad \begin{pmatrix}\begin{array}{c}(v_j : G_j)_{j\in U_i\cup V_i}, ((u_j : A_j)^\circ)_{j=\hat{l}+1}^{\widetilde{l}} \\ \widetilde{\mathscr{G}_i} \\ \dfrac{(\widetilde{C}_j)_{j\in K_i''} \to \widetilde{H}_i \quad (\widetilde{u}_j : \widetilde{C}_j)_{j\in K_i''}}{\widetilde{H}_i}\to E\end{array}\end{pmatrix}_{i=1}^{\widetilde{q}}}{A_{\widetilde{l}+1}^l \to B}\to E}{A_{\hat{l}+1}^l \to B}\to I, u_{\hat{l}+1}^{\widetilde{l}}}{(G_i)_{i\in P^+} \to A_{\hat{l}+1}^l \to B}\to I, (v_i)_{i\in P^+}
}{
\dfrac{C}{\widetilde{C}_1^{\widetilde{k}} \to C}\to I, \widetilde{u}_1^{\widetilde{k}}
}\to E
$$

where $\hat{\mathscr{C}}_1^k$ is as in (49) and i_1 is defined immediately above (55). Since $\widetilde{\mathscr{G}_i}$ satisfies condition (I3), $\hat{\mathscr{E}}_1$ normalizes by a sequence of non-erasing β-reduction steps. We can show

(79) $|\hat{\mathscr{D}}_1|_\beta, (\mathscr{D}_j')_{j\in T^-}, (\mathscr{B}_j)_{j\in V}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_1$ via $|\hat{\mathscr{E}}_1|_\beta$.

That $\hat{\mathscr{E}}_1$ satisfies condition (I4) of Definition 16 can be checked using (F) and (I). It remains to show $\hat{\mathscr{E}}_1[\hat{\mathscr{D}}_1/\check{z}_1, (\mathscr{D}_j'/w_j)_{j\in T^-}, (\mathscr{B}_j/v_j)_{j\in V}] \twoheadrightarrow_\beta \widetilde{\mathscr{D}}_1$.

$$\hat{\mathscr{E}}_1[\hat{\mathscr{D}}_1/\check{z}_1, (\mathscr{D}_j'/w_j)_{j\in T^-}, (\mathscr{B}_j/v_j)_{j\in V}]$$

=

$$
\dfrac{
\dfrac{
\begin{matrix}\Gamma'_1\cup\bigcup_{i\in P^+}\Gamma''_i\\ \widehat{\mathscr{D}}_1\\ C_1^k\to((G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B)\to C\end{matrix}
\quad
\begin{matrix}\bigcup_{j\in T^-}\Gamma'_j,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widehat{\mathscr{C}}_1^k[(\mathscr{D}'_j/w_j)_{j\in T^-}]\\ C_1^k\end{matrix}
}{((G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B)\to C}\to E
\qquad
\dfrac{
\dfrac{
\dfrac{
\widetilde u_{i_1}:\widetilde H_1^{\widetilde q}\to A^l_{\widetilde l+1}\to B
\quad
\left(\dfrac{\begin{matrix}(v_j:G_j)_{j\in U_i},\bigcup_{j\in V_i}\Gamma''_j,((u_j:A_j)^\circ)_{j=\hat l+1}^{\widetilde l}\\ \widetilde{\mathscr{G}_i}[(\mathscr{B}_j/v_j)_{j\in V_i}]\\ (\widetilde C_j)_{j\in K''_i}\to\widetilde H_i\quad(\widetilde u_j:\widetilde C_j)_{j\in K''_i}\end{matrix}}{\widetilde H_i}\to E\right)_{i=1}^{\widetilde q}
}{A^l_{\widetilde l+1}\to B}\to E
}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}
}{(G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B}\to I,(v_i)_{i\in P^+}
}{\dfrac{C}{\widetilde C_1^k\to C}\to I,\widetilde u_1^k}\to E
$$

$\twoheadrightarrow_\beta$ (by (10))

$$
\dfrac{
\dfrac{
\begin{matrix}\Gamma'_1\\ \mathscr{D}'_1\\ C_1^k\to(A^l_1\to B)\to C\end{matrix}
\quad
\begin{matrix}\bigcup_{j\in T^-}\Gamma'_j,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widehat{\mathscr{C}}_1^k[(\mathscr{D}'_j/w_j)_{j\in T^-}]\\ C_1^k\end{matrix}
}{(A^l_1\to B)\to C}\to E
\qquad
\dfrac{
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\dfrac{
\dfrac{
\widetilde u_{i_1}:\widetilde H_1^{\widetilde q}\to A^l_{\widetilde l+1}\to B
\quad
\left(\dfrac{\begin{matrix}(v_j:G_j)_{j\in U_i},\bigcup_{j\in V_i}\Gamma''_j,((u_j:A_j)^\circ)_{j=\hat l+1}^{\widetilde l}\\ \widetilde{\mathscr{G}_i}[(\mathscr{B}_j/v_j)_{j\in V_i}]\\ (\widetilde C_j)_{j\in K''_i}\to\widetilde H_i\quad(\widetilde u_j:\widetilde C_j)_{j\in K''_i}\end{matrix}}{\widetilde H_i}\to E\right)_{i=1}^{\widetilde q}
}{A^l_{\widetilde l+1}\to B}\to E
}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}
}{(G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B}\to I,(v_i)_{i\in P^+}
\quad
\left(\begin{matrix}\Gamma''_i,((u_j:A_j)^\circ)_{j=1}^{\hat l}\\ \mathscr{B}_i\\ G_i\end{matrix}\right)_{i\in P^+}
}{A^l_{\hat l+1}\to B}\to E
}{A^l_1\to B}\to I,u_1^{\hat l}
}{\dfrac{C}{\widetilde C_1^k\to C}\to I,\widetilde u_1^k}\to E
$$

$\twoheadrightarrow_\beta$ (by (53))

$$
\dfrac{
\begin{matrix}\widetilde\Gamma'_1,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widetilde{\mathscr{D}}'_1\\ (A^l_1\to B)\to C\end{matrix}
\qquad
\dfrac{
\dfrac{
\dfrac{
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\dfrac{
\widetilde u_{i_1}:\widetilde H_1^{\widetilde q}\to A^l_{\widetilde l+1}\to B
\quad
\left(\dfrac{\begin{matrix}(v_j:G_j)_{j\in U_i},\bigcup_{j\in V_i}\Gamma''_j,((u_j:A_j)^\circ)_{j=\hat l+1}^{\widetilde l}\\ \widetilde{\mathscr{G}_i}[(\mathscr{B}_j/v_j)_{j\in V_i}]\\ (\widetilde C_j)_{j\in K''_i}\to\widetilde H_i\quad(\widetilde u_j:\widetilde C_j)_{j\in K''_i}\end{matrix}}{\widetilde H_i}\to E\right)_{i=1}^{\widetilde q}
}{A^l_{\widetilde l+1}\to B}\to E
}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}
}{(G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B}\to I,(v_i)_{i\in P^+}
\quad
\left(\begin{matrix}\Gamma''_i,((u_j:A_j)^\circ)_{j=1}^{\hat l}\\ \mathscr{B}_i\\ G_i\end{matrix}\right)_{i\in P^+}
}{A^l_{\hat l+1}\to B}\to E
}{A^l_1\to B}\to I,u_1^{\hat l}
}{\dfrac{C}{\widetilde C_1^k\to C}\to I,\widetilde u_1^k}\to E
$$

$\twoheadrightarrow_\beta$

$$
\dfrac{\dfrac{\begin{array}{c}\widetilde\Gamma'_1,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widetilde{\mathscr D}'_1\\ (A^l_1\to B)\to C\end{array}\qquad \dfrac{\dfrac{\dfrac{\widetilde u_{i_1}:\widetilde H^{\widetilde q}_1\to A^l_{\widetilde l+1}\to B\qquad \left(\dfrac{\begin{array}{c}\bigcup_{j\in U_i}\Gamma''_j,((u_j:A_j)^\circ)^{\hat l}_{j=1},\bigcup_{j\in V_i}\Gamma''_j,((u_j:A_j)^\circ)^{\widetilde l}_{j=\hat l+1}\\ \widetilde{\mathscr G}_i[(\mathscr B_j/v_j)_{j\in V_i}][(\mathscr B_j/v_j)_{j\in U_i}]\\ (\widetilde C_j)_{j\in K''_i}\to\widetilde H_i\end{array}\qquad(\widetilde u_j:\widetilde C_j)_{j\in K''_i}}{\widetilde H_i}\to E\right)^{\widetilde q}_{i=1}}{A^l_{\widetilde l+1}\to B}\to E}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}}{A^l_1\to B}\to I,u^{\hat l}_1}{C}\to E}{\widetilde C^k_1\to C}\to I,\widetilde u^k_1
$$

$\twoheadrightarrow_\beta$ (by (77.ii.a))

$$
\dfrac{\dfrac{\begin{array}{c}\widetilde\Gamma'_1,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widetilde{\mathscr D}'_1\\ (A^l_1\to B)\to C\end{array}\qquad \dfrac{\dfrac{\dfrac{\widetilde u_{i_1}:\widetilde H^{\widetilde q}_1\to A^l_{\widetilde l+1}\to B\qquad \left(\dfrac{\begin{array}{c}\widetilde\Gamma''_{1,i},((u_j:A_j)^\circ)^{\widetilde l}_{j=1}\\ \widetilde{\mathscr B}_i\\ (\widetilde C_j)_{j\in K''_i}\to\widetilde H_i\end{array}\qquad(\widetilde u_j:\widetilde C_j)_{j\in K''_i}}{\widetilde H_i}\to E\right)^{\widetilde q}_{i=1}}{A^l_{\widetilde l+1}\to B}\to E}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}}{A^l_1\to B}\to I,u^{\hat l}_1}{C}\to E}{\widetilde C^k_1\to C}\to I,\widetilde u^k_1
$$

$\twoheadrightarrow_\beta$ (by (59))

$$
\dfrac{\dfrac{\begin{array}{c}\widetilde\Gamma'_1,(\widetilde u_j:\widetilde C_j)_{j\in K'}\\ \widetilde{\mathscr D}'_1\\ (A^l_1\to B)\to C\end{array}\qquad \dfrac{\dfrac{\dfrac{\widetilde u_{i_1}:\widetilde H^{\widetilde q}_1\to A^l_{\widetilde l+1}\to B\qquad \left(\begin{array}{c}\widetilde\Gamma''_{1,i},(\widetilde u_j:\widetilde C_j)_{j\in K''_i},((u_j:A_j)^\circ)^{\widetilde l}_{j=1}\\ \widetilde{\mathscr H_i}\\ \widetilde H_i\end{array}\right)^{\widetilde q}_{i=1}}{A^l_{\widetilde l+1}\to B}\to E}{A^l_{\hat l+1}\to B}\to I,u^{\widetilde l}_{\hat l+1}}{A^l_1\to B}\to I,u^{\hat l}_1}{C}\to E}{\widetilde C^k_1\to C}\to I,\widetilde u^k_1
$$

= (by (41) and (55))

$$\widetilde{\mathscr D}_1$$

The output $\mathscr D^m_1,\mathscr D_0$ of the new method is the result of applying the pruning procedure to $(\check{\mathscr D}_i\colon\check\Gamma_i\Rightarrow E_i)^{\check m}_{i=1},\check{\mathscr D}_0\colon(\check z_i:\check E_i)^{\check m}_{i=1},\Delta\Rightarrow C$, where

$$
\begin{aligned}
\check{\mathscr D}^{\check m}_1&=|\hat{\mathscr D}_1|_\beta,(\mathscr D'_i)_{i\in N},(\mathscr B_i)_{i\in P^-},\\
(\check z_i:\check E_i)^{\check m}_{i=1}&=\check z_1:C^k_1\to((G_i)_{i\in P^+}\to A^l_{\hat l+1}\to B)\to C,(w_i:F_i)_{i\in N},(v_i:G_i)_{i\in P^-},
\end{aligned}
$$

as described in Case 2.2.1.2 of the new method. Let $n_1,\dots,n_r$ and $p_1,\dots,p_s$ list the elements of N and P^-, respectively, in increasing order, so that $\check m=1+r+s$.

Let $\mu(i)$ and $\mathscr{M}_i$ be as in the description of prune$(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. We have $\mu(1) = 1$ and $\mathscr{D}_1 = \check{\mathscr{D}}_1 = |\hat{\mathscr{D}}_1|_\beta$ is an $\varnothing$-interpolant to itself via $\mathscr{M}_1$. For $i = 1, \dots, r$, $\mathscr{D}_{\mu(1+i)}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_{1+i} = \mathscr{D}'_{n_i}$ via $\mathscr{M}_{1+i}$. For $i = 1, \dots, s$, $\mathscr{D}_{\mu(1+r+i)}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_{1+r+i} = \mathscr{B}_{p_i}$ via $\mathscr{M}_{1+r+i}$. We define subsets $S_1^{\widetilde{m}}$ of $\{1, \dots, m\}$ and deductions $\mathscr{E}_i \colon (z_j : E_j)_{j \in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ as follows:

$$S_i = \begin{cases} \{1\} \cup \{\, \mu(1+j) \mid n_j \in T^- \,\} \cup \{\, \mu(1+r+j) \mid p_j \in V \,\} & \text{if } i = 1, \\ \{\, \mu(1+j) \mid n_j \in S'_i \,\} & \text{if } i \in M' - \{1\}, \\ \{\, \mu(1+r+j) \mid p_j \in S''_i \,\} & \text{if } i \in M'' - M' - \{1\}. \end{cases}$$

$$\mathscr{E}_i = \begin{cases} |\hat{\mathscr{E}}_1[\mathscr{M}_1/\check{z}_1, (\mathscr{M}_{1+j}/w_{n_j})_{n_j \in T^-}, (\mathscr{M}_{1+r+j}/v_{p_j})_{p_j \in V}]|_\beta & \text{if } i = 1, \\ |\mathscr{E}'_i[(\mathscr{M}_{1+j}/w_{n_j})_{n_j \in S'_i}]|_\beta & \text{if } i \in M' - \{1\}, \\ |\mathscr{E}''_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j \in S''_i}]|_\beta & \text{if } i \in M'' - M' - \{1\}. \end{cases}$$

($\hat{\mathscr{E}}_1, \mathscr{E}'_i, \mathscr{E}''_i$ are given in (78), (48), and (77), respectively.) We show that $S_1^{\widetilde{m}}$ and $\mathscr{E}_1^{\widetilde{m}}$ satisfy conditions 2–4.

Condition 2 follows from (79), (48.ii.b), and (77.ii.b), using the property of $\mathscr{M}_i$ mentioned above. Condition 4 easily follows from (23).

It remains to prove condition 3. From (35) and (78), we see that

$$\widetilde{\mathscr{D}}_0[(\mathscr{E}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] =_\beta \widetilde{\mathscr{D}}_0[\hat{\mathscr{E}}_1[\mathscr{M}_1/\check{z}_1, (\mathscr{M}_{1+j}/w_{n_j})_{n_j \in T^-}, (\mathscr{M}_{1+r+j}/v_{p_j})_{p_j \in V}]/\widetilde{z}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i=2}^{\widetilde{m}}]$$

$$\twoheadrightarrow_\beta \quad \dfrac{\dfrac{\begin{array}{c} z_1 : C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C \\ \mathscr{M}_1 \\ C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C \end{array} \qquad \widetilde{\mathscr{L}}_1^k}{((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C} \to E \qquad \widetilde{\mathscr{R}}}{C} \to E$$

where

$$\widetilde{\mathscr{L}}_1^k = \begin{array}{c} (z_h : E_h)_{h \in \{\, \mu(1+j) \mid n_j \in T^- \,\} \cup \bigcup_{i \in M'} S_i}, \Delta' \\ \hat{\mathscr{C}}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j \in T^-}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_i/\widetilde{z}_i)_{i \in M_j}]/\widetilde{u}_j)_{j \in K'}] \\ C_1^k \end{array}$$

$$\widetilde{\mathscr{R}} = \dfrac{\begin{array}{c} (z_j : E_j)_{j \in \bigcup_{i \in M_{i_1}} S_i}, \widetilde{\Delta}_{i_1} \\ \widetilde{\mathscr{C}}_{i_1}[(\mathscr{E}_i/\widetilde{z}_i)_{i \in M_{i_1}}] \\ \widetilde{H}_1^{\widetilde{q}} \to A^l_{\widetilde{l}+1} \to B \end{array} \quad \left(\dfrac{\begin{array}{c} (v_j : G_j)_{j \in U_i}, (z_{\mu(1+r+j)} : E_{\mu(1+r+j)})_{p_j \in V_i}, ((u_j : A_j)^\circ)_{j=\hat{l}+1}^{\widetilde{l}} \\ \widetilde{\mathscr{G}}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j \in V_i}] \\ (\widetilde{C}_j)_{j \in K''_i} \to \widetilde{H}_i \end{array} \quad \left(\begin{array}{c} (z_g : E_g)_{g \in \bigcup_{h \in M_j} S_h}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h \in M_j}] \\ \widetilde{C}_j \end{array} \right)_{j \in K''_i}}{\widetilde{H}_i} \to E \right)_{i=1}^{\widetilde{q}}}{\dfrac{\dfrac{A^l_{\widetilde{l}+1} \to B}{A^l_{\hat{l}+1} \to B} \to I, u^{\widetilde{l}}_{\hat{l}+1}}{(G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B} \to I, (v_i)_{i \in P^+}} \to E$$

Since $\mathscr{D}_0 = |\check{\mathscr{D}}_0[(\mathscr{M}_i/\check{z}_i)_{i=1}^{\check{m}}]|_\beta$, where $\check{\mathscr{D}}_0$ is given in (11), $\mathscr{D}_0$ is the normal form of

$$\dfrac{\dfrac{\begin{array}{c} z_1 : C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C \\ \mathscr{M}_1 \\ C_1^k \to ((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C \end{array} \qquad \mathscr{L}_1^k}{((G_i)_{i \in P^+} \to A^l_{\hat{l}+1} \to B) \to C} \to E \qquad \mathscr{R}}{C} \to E$$

where

$$\mathscr{L}^k_1 = \begin{array}{c} (z_{\mu(1+i)} : E_{\mu(1+i)})_{n_i\in N}, \Delta' \\ \mathscr{C}^k_1[(\mathscr{M}_{1+i}/w_{n_i})_{n_i\in N}] \\ C^k_1 \end{array}$$

$$\mathscr{R} = \begin{array}{c} (v_i : G_i)_{i\in P^+}, (z_{\mu(1+r+i)} : E_{\mu(1+r+i)})_{p_i\in P^-}, (u_j : A_j)^l_{j=\hat{l}+1}, \Delta'' \\ \dfrac{\begin{array}{c}\hat{\mathscr{B}}_0[(\mathscr{M}_{1+r+i}/v_{p_i})_{p_i\in P^-}] \\ B\end{array}}{(G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B} \to I, (v_i)_{i\in P^+}, u^l_{\hat{l}+1} \end{array}$$

By (56) and (60),

$$\widetilde{\mathscr{R}}$$

$$=_\beta \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}(z_j : E_j)_{j\in\bigcup_{i\in M_{i_1}} S_i}, \widetilde{\Delta}_{i_1} \\ \widetilde{\mathscr{C}_{i_1}}[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_{i_1}}] \\ \widetilde{H}^{\tilde{q}}_1 \to A^l_{\tilde{l}+1} \to B\end{array} \quad \left(\dfrac{\begin{array}{c}(v_j : G_j)_{j\in U_i}, (z_{\mu(1+r+j)} : E_{\mu(1+r+j)})_{p_j\in V_i}, ((u_j : A_j)^\circ)^{\tilde{l}}_{j=\hat{l}+1} \\ \widetilde{\mathscr{G}_i}[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}] \\ (\widetilde{C}_j)_{j\in K''_i} \to \widetilde{H}_i\end{array} \quad \left(\begin{array}{c}(z_g : E_g)_{g\in\bigcup_{h\in M_j} S_h}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}_j}[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}] \\ \widetilde{C}_j\end{array}\right)_{j\in K''_i}}{\widetilde{H}_i} \to E\right)^{\tilde{q}}_{i=1}}{A^l_{\tilde{l}+1} \to B} \to E \quad (u_j : A_j)^l_{\tilde{l}+1}}{B} \to E}{A^l_{\hat{l}+1} \to B} \to I, u^l_{\hat{l}+1}}{(G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B} \to I, (v_i)_{i\in P^+}$$

$$=_\beta \begin{array}{c} (v_i : G_i)_{i\in P^+}, (z_h : E_h)_{h\in\{\mu(1+r+j) \mid p_j\in V\}\cup\bigcup_{i\in M''} S_i}, (u_j : A_j)^l_{j=\hat{l}+1}, \Delta'' \\ \dfrac{\dfrac{\begin{array}{c}\widetilde{\mathscr{B}}_0[(\widetilde{\mathscr{G}_i}[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)^{\tilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (u_j : A_j/\widetilde{y}_j)^l_{j=\tilde{l}+1}] \\ B\end{array}}{A^l_{\hat{l}+1} \to B} \to I, u^l_{\hat{l}+1}}{(G_i)_{i\in P^+} \to A^l_{\hat{l}+1} \to B} \to I, (v_i)_{i\in P^+} \end{array}$$

So it suffices to show

(80) $\hat{\mathscr{C}}^k_1[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\widetilde{\mathscr{C}_j}[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}] =_\beta \mathscr{C}^k_1[(\mathscr{M}_{1+i}/w_{n_i})_{n_i\in N}]$,

and

(81)
$\widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)^{\tilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\tilde{l}+1}] =_\beta \mathscr{B}_0[(\mathscr{M}_{1+r+i}/v_{p_i})_{p_i\in P^-}]$.

We first show (80). By (47), (48.ii.a), and condition 2, we get

(82) $(\mathscr{D}'_j)_{j\in T}, (\mathscr{D}_j)_{j\in\bigcup_{i\in M'} S_i}$ is an interpolant to $\mathscr{D}'$ with respect to $(\Gamma'; \Delta')$ via the normal form of $\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}]$: $(w_j : F_j)_{j\in T}, (z_j : E_j)_{j\in\bigcup_{i\in M'} S_i}, \Delta' \Rightarrow (A^l_1 \to B) \to C$.

Applying the induction hypothesis again to $\mathscr{D}'$ with respect to (82) and noting Lemma 32, we obtain elements $(\hat{\rho}(j))_{j\in\bigcup_{i\in M'} S_i}$ of $\{1, \ldots, n\}$ and deductions $(\mathscr{I}_j\colon w_j\colon F_j \Rightarrow F_j)_{j\in T}, (\mathscr{P}_j\colon w_{\hat{\rho}(j)} : F_{\hat{\rho}(j)} \Rightarrow E_j)_{j\in\bigcup_{i\in M'} S_i}$ such that

(83) i. $T \cup \{\hat{\rho}(j) \mid j \in \bigcup_{i\in M'} S_i\} = \{1, \ldots, n\}$;

ii. a. $\mathscr{D}'_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I}_j$ for each $j \in T$;

b. $\mathscr{D}'_{\hat{\rho}(j)}$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{P}_j$ for each $j \in \bigcup_{i\in M'} S_i$;

iii. $\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}][(\mathscr{I}_j/w_j)_{j\in T}, (\mathscr{P}_j/z_j)_{j\in\bigcup_{i\in M'} S_i}] \twoheadrightarrow_\beta \mathscr{D}'_0$.

By (48.iv.a),

(84) $$\mathscr{F}[(\mathscr{I}_j/w_j)_{j\in T}] \twoheadrightarrow_\beta \mathscr{F}.$$

We have

$$\begin{aligned}
&\widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M'}][(\mathscr{I}_j/w_j)_{j\in T}, (\mathscr{P}_j/z_j)_{j\in\bigcup_{i\in M'} S_i}]\\
&= \widetilde{\mathscr{D}}'_0[\mathscr{F}[(\mathscr{I}_j/w_j)_{j\in T}]/\widetilde{w}_1, (\mathscr{E}_i[(\mathscr{P}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M'}]\\
&\twoheadrightarrow_\beta \widetilde{\mathscr{D}}'_0[\mathscr{F}/\widetilde{w}_1, (\mathscr{E}_i[(\mathscr{P}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M'}] \quad \text{by (84)}
\end{aligned}$$

$$= \dfrac{\dfrac{\dfrac{w_1 : C_1^k \to (A_1^l \to B) \to C \qquad \begin{array}{c}(w_j : F_j)_{j\in T^-}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'}\\ \hat{\mathscr{C}}_1^k \\ C_1^k\end{array}}{(A_1^l \to B) \to C}\to E}{(\widetilde{C}_j)_{j\in K'} \to (A_1^l \to B) \to C}\to I, (\widetilde{u}_j)_{j\in K'} \qquad \left(\begin{array}{c}(w_{\hat{\rho}(h)} : F_{\hat{\rho}(h)})_{h\in\bigcup_{i\in M_j} S_i}, \widetilde{\Delta}_j\\ \widetilde{\mathscr{C}}_j[(\mathscr{E}_i[(\mathscr{P}_h/z_h)_{h\in S_i}]/\widetilde{z}_i)_{i\in M_j}]\\ \widetilde{C}_j\end{array}\right)_{j\in K'}}{(A_1^l \to B) \to C}\to E$$

by (46) and (49)

$$\twoheadrightarrow_\beta \dfrac{w_1 : C_1^k \to (A_1^l \to B) \to C \qquad \begin{array}{c}(w_h : F_h)_{h\in T^-\cup\{\hat{\rho}(j) \mid j\in\bigcup_{i\in M'} S_i\}}, \Delta'\\ \hat{\mathscr{C}}_1^k[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_i[(\mathscr{P}_h/z_h)_{h\in S_i}]/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}]\\ C_1^k\end{array}}{(A_1^l \to B) \to C}\to E$$

Therefore, by (6) and (83.iii),

(85) $$\hat{\mathscr{C}}_1^k[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_i[(\mathscr{P}_h/z_h)_{h\in S_i}]/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}] \twoheadrightarrow_\beta \mathscr{C}_1^k,$$

which implies that

(86) $$T^- \cup \{\, \hat{\rho}(j) \mid j \in \bigcup_{i\in M'} S_i \,\} = N.$$

Let $(\rho(j))_{j\in\bigcup_{i\in M'} S_i}$ be such that $\hat{\rho}(j) = n_{\rho(j)}$ for each $j \in \bigcup_{i\in M'} S_i$. By (86),

(87) $$T^- \cup \{\, n_{\rho(j)} \mid j \in \bigcup_{i\in M'} S_i \,\} = N.$$

By (25), (83.ii.b) implies that for $j \in \bigcup_{i\in M'} S_i$, $\mu(1+\rho(j)) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{D}'_{n_{\rho(j)}}$ via $\mathscr{M}_{1+\rho(j)}\colon z_j : E_j \Rightarrow F_{n_{\rho(j)}}$. It follows that $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{P}_j[\mathscr{M}_{1+\rho(j)}/w_{n_{\rho(j)}}]\colon z_j{:}E_j \Rightarrow E_j$. Hence by condition 4,

(88) $$\mathscr{E}_i[(\mathscr{P}_j[\mathscr{M}_{1+\rho(j)}/w_{n_{\rho(j)}}]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M'.$$

Now we can show (80) as follows:

$$\begin{aligned}
&\hat{\mathscr{C}}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}]\\
&=_\beta \hat{\mathscr{C}}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_i[(\mathscr{P}_h[\mathscr{M}_{1+\rho(h)}/w_{n_{\rho(h)}}]/z_h)_{h\in S_i}]/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}] \quad \text{by (88)}\\
&= \hat{\mathscr{C}}_1^k[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_i[(\mathscr{P}_h/z_h)_{h\in S_i}]/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}][(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in N}] \quad \text{by (87)}\\
&\twoheadrightarrow_\beta \mathscr{C}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in N}] \quad \text{by (85).}
\end{aligned}$$

We now turn to (81). By (61), (77.ii.a), and condition 2,

(89) $(\mathscr{B}_j)_{j\in P^+\cup V}, (\mathscr{D}_j)_{j\in\bigcup_{i\in M''} S_i}, (u_j : A_j)^l_{j=\hat{l}+1}$ is an interpolant to $\mathscr{B}$ with respect to $(\Gamma'', ((u_j{:}A_j)^\circ)^l_{j=1}; \Delta'')$ via the normal form of $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}] \colon (v_j : G_j)_{j\in P^+\cup V}, (z_j : E_j)_{j\in\bigcup_{i\in M''} S_i}, (v_{a_j} : A_j)^l_{j=\widetilde{l}+1} \Rightarrow B$.

Applying the induction hypothesis again to $\mathscr{B}$ with respect to (89) and noting Lemma 32, we obtain elements $(\tau(j))_{j\in\bigcup_{i\in M''} S_i}$ of $\{1,\dots,s\}$ and deductions $(\mathscr{I}_j \colon v_j : G_j \Rightarrow G_j)_{j\in P^+\cup V}, (\mathscr{T}_j \colon v_{p_{\tau(j)}} : G_{p_{\tau(j)}} \Rightarrow E_j)_{j\in\bigcup_{i\in M''} S_i}$ such that

(90) i. $V \cup \{\, p_{\tau(j)} \mid j \in \bigcup_{i\in M''} S_i \,\} = P^-$;

ii. a. $\mathscr{B}_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I}_j$ for $j \in P^+ \cup V$;

b. $\mathscr{B}_{p_{\tau(j)}}$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{T}_j$ for $j \in \bigcup_{i\in M''} S_i$;

iii. $\widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j}{:}A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}][(\mathscr{I}_j/v_j)_{j\in P^+\cup V}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}] \twoheadrightarrow_\beta \mathscr{B}_0$.

By (25), (90.ii.b) implies that for $j \in \bigcup_{i\in M''} S_i$, $\mu(1 + r + \tau(j)) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{B}_{p_{\tau(j)}}$ via $\mathscr{M}_{1+r+\tau(j)} \colon z_j : E_j \Rightarrow G_{p_{\tau(j)}}$. It follows that $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{T}_j[\mathscr{M}_{1+r+\tau(j)}/v_{p_{\tau(j)}}] \colon z_j : E_j \Rightarrow E_j$. Hence by condition 4,

$$\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{1+r+\tau(j)}/v_{p_{\tau(j)}}]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M''. \tag{91}$$

Also, by (90.ii.a) and (77.iv.a),

$$\mathscr{G}_i[(\mathscr{I}_{p_j}/v_{p_j})_{p_j\in U_i\cup V_i}] \twoheadrightarrow_\beta \mathscr{G}_i \quad \text{for } i = 1,\dots,\widetilde{q}. \tag{92}$$

Now we can show (81) as follows:

$$\begin{aligned}
&\widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}]\\
&=_\beta \widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{I}_{p_j}/v_{p_j})_{p_j\in U_i\cup V_i}][(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}]/\widetilde{v}_i)^{\widetilde{q}}_{i=1},\\
&\qquad (\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{1+r+\tau(j)}/v_{p_{\tau(j)}}]/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}] \quad \text{by (91) and (92)}\\
&= \widetilde{\mathscr{B}}_0[(\mathscr{G}_i[(\mathscr{I}_{p_j}/v_{p_j})_{p_j\in U_i\cup V_i}]/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\mathscr{E}_i[(\mathscr{T}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}]\\
&\qquad [(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P^-}] \quad \text{by (90.i)}\\
&= \widetilde{\mathscr{B}}_0[(\mathscr{G}_i/\widetilde{v}_i)^{\widetilde{q}}_{i=1}, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}, (v_{a_j} : A_j/\widetilde{y}_j)^l_{j=\widetilde{l}+1}][(\mathscr{I}_{p_j}/v_{p_j})_{p_j\in P^+\cup V}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}]\\
&\qquad [(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P^-}]\\
&\twoheadrightarrow_\beta \mathscr{B}_0[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P^-}] \quad \text{by (90.iii)}
\end{aligned}$$

Case 2.2b.2. The main branch of $\mathscr{D}''$ leads to an assumption belonging to Γ'' or to some $u_j : A_j$, i.e., Case 2.2.2 of the description of the new method applies. Then (D) and (43) imply that the main branch of $\widetilde{\mathscr{D}}''_1$ must lead to an assumption belonging to $\widetilde{\Gamma}''_1$ or to some $u_j : A_j$, and $\widetilde{\mathscr{D}}''_1$ must have the following form:

$$\widetilde{\mathscr{D}}''_1 = \dfrac{\dfrac{\begin{array}{c}\widetilde{\Gamma}''_1, (\widetilde{u}_i : \widetilde{C}_i)_{i\in K''}, ((u_j : A_j)^\circ)^l_{j=1}\\ \widetilde{\mathscr{D}}''^{-}_1\end{array}}{B}}{A^l_1 \to B}\,{\to}I, u^l_1 \tag{93}$$

where $\widetilde{\mathscr{D}}''^{-}_1$ does not end in $\to I$. Since $\widetilde{\mathscr{D}}_1$ satisfies condition (I3) of Definition 16, $\widetilde{\mathscr{D}}''^{-}_1$ must satisfy the following condition:

(J) Every maximal path in $\widetilde{\mathscr{D}}_1''^{-}$ that starts inside the endformula B or some $\widetilde{u}_j : \widetilde{C}_j$ must end inside an assumption belonging to $\widetilde{\Gamma}_1''$ or some $u_j : A_j$.

From (43) we obtain

(94) $$\widetilde{\mathscr{D}}_1''^{-}[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K''}] \twoheadrightarrow_\beta \mathscr{B}.$$

Let

(95) $$\widetilde{\mathscr{B}} = \dfrac{\begin{array}{c}\widetilde{\Gamma}_1'', (\widetilde{u}_i : \widetilde{C}_i)_{i\in K''}, ((u_j : A_j)^\circ)_{j=1}^{l} \\ \widetilde{\mathscr{D}}_1''^{-} \\ B\end{array}}{(\widetilde{C}_i)_{i\in K''} \to B} \to I, (\widetilde{u}_i)_{i\in K''}$$

(96) $$\widetilde{\mathscr{B}}_0 = \dfrac{\widetilde{v}_1 : (\widetilde{C}_i)_{i\in K''} \to B \quad \left(\begin{array}{c}(\widetilde{z}_j : \widetilde{E}_j)_{j\in M_i}, \widetilde{\Delta}_i \\ \widetilde{\mathscr{C}}_i \\ \widetilde{C}_i\end{array}\right)_{i\in K''}}{B} \to E$$

where $\widetilde{\mathscr{C}}_i$ are as in (35). We show

(97) $\widetilde{\mathscr{B}}, (\widetilde{\mathscr{D}}_j)_{j\in M''}$ is an interpolant to $\mathscr{B}$ with respect to $(\Gamma'', ((u_j : A_j)^\circ)_{j=1}^{l}; \Delta'')$ via $\widetilde{\mathscr{B}}_0$.

Firstly,

$$\begin{aligned}\widetilde{\mathscr{B}}_0[\widetilde{\mathscr{B}}/\widetilde{v}_1, (\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M''}] &\twoheadrightarrow_\beta \widetilde{\mathscr{D}}_1''^{-}[(\widetilde{\mathscr{C}}_i[(\widetilde{\mathscr{D}}_j/\widetilde{z}_j)_{j\in M_i}]/\widetilde{u}_i)_{i\in K''}] \\ &\twoheadrightarrow_\beta \mathscr{B} \quad \text{by (94).}\end{aligned}$$

Secondly, $\widetilde{\mathscr{B}}$ satisfies condition (I3) of Definition 16 by (J). Finally, the property (D) ensures that $\widetilde{\mathscr{B}}_0$ satisfies condition (I4) of Definition 16. So we have shown (97).

By the induction hypothesis, we have subsets $W, (S_i'')_{i\in M''}$ of $\{1, \ldots, p\}$ and deductions $\mathscr{G} : (v_j : G_j)_{j\in W} \Rightarrow (\widetilde{C}_i)_{i\in K''} \to B, (\mathscr{E}_i'' : (v_j : G_j)_{j\in S_i''} \Rightarrow \widetilde{E}_i)_{i\in M''}$ such that

(98) i. $W \cup \bigcup_{i\in M''} S_i'' = \{1, \ldots, p\}$;

ii. a. $(\mathscr{B}_j)_{j\in W}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{B}}$ via $\mathscr{G}$;

b. for each $i \in M''$, $(\mathscr{B}_j)_{j\in S_i''}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_i$ via $\mathscr{E}_i''$;

iii. $\widetilde{\mathscr{B}}_0[\mathscr{G}/\widetilde{v}_1, (\mathscr{E}_i''/\widetilde{z}_i)_{i\in M''}] \twoheadrightarrow_\beta \mathscr{B}_0$;

iv. a. for $j \in W$, $\mathscr{G}$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$;

b. for $i \in M''$ and for $j \in S_i''$, $\mathscr{E}_i''$ is long for $\mathscr{B}_j$ with respect to $v_j : G_j$.

By (98.ii.a), (98.iv.a), and part 2 of Lemma 44, $\widetilde{\mathscr{B}}$ and $\mathscr{G}$ have identical final blocks of applications of $\to I$. Since $\widetilde{\mathscr{B}}$ satisfies condition (I3) of Definition 16, it follows from (98.ii.a) and Lemma 21 that $\mathscr{G}$ also satisfies condition (I3). By (12), (95), and (98.iii), then, $\mathscr{G}$ must be of the following form:

(99) $$\mathscr{G} = \dfrac{\dfrac{v_1 : H_1^q \to B \quad \left(\begin{array}{c}(v_j : G_j)_{j\in W_i}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K_i''} \\ \hat{\mathscr{H}}_i \\ H_i\end{array}\right)_{i=1}^{q}}{B} \to E}{(\widetilde{C}_j)_{j\in K''} \to B} \to I, (\widetilde{u}_j)_{j\in K''}$$

where

(100)
$$\begin{aligned}\{1\} \cup W_1 \cup \cdots \cup W_q &= W,\\ K''_1 \cup \cdots \cup K''_q &= K'',\\ W_i \cup \bigcup_{j \in K''_i} \bigcup_{h \in M_j} S''_h &= P_i.\end{aligned}$$

Since $\mathscr{G}$ satisfies condition (I4) of Definition 16, each $\hat{\mathscr{H}}_i$ must satisfy the following condition:

(K) Every maximal path in $\hat{\mathscr{H}}_i$ that starts inside the endformula or some $v_j : G_j$ must end inside some $\widetilde{u}_j : \widetilde{C}_j$.

Since $\mathscr{G}[(\mathscr{B}_i/v_i)_{i\in W}] \twoheadrightarrow_\beta \widetilde{\mathscr{B}}$ by (98.ii.a),

(101)
$$\dfrac{\begin{matrix}\Gamma''_1, ((u_j : A_j)^\circ)_{j=1}^l\\ \mathscr{B}_1\\ H_1^q \to B\end{matrix} \qquad \left(\begin{matrix}\bigcup_{j\in W_i} \Gamma''_j, ((u_j : A_j)^\circ)_{j=1}^l, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}\\ \hat{\mathscr{H}}_i[(\mathscr{B}_j/v_j)_{j\in W_i}]\\ H_i\end{matrix}\right)_{i=1}^q}{B} \to E \qquad \twoheadrightarrow_\beta \widetilde{\mathscr{D}}''^{-}_1$$

By (98.ii.b), $S''_i \subseteq P^-$ for $i \in M''$, so we have

$$P_i^+ \subseteq W_i.$$

Let

$$V_i = W_i - P_i^+,$$

so that

(102)
$$W_i = V_i \cup P_i^+,$$

(103)
$$P_i^- = V_i \cup \bigcup_{j\in K''_i} \bigcup_{h\in M_j} S''_h.$$

Let $V = \bigcup_{i=1}^q V_i$ and let $U = W - V$. We have

$$\begin{aligned}P^- &= V \cup \bigcup_{i\in M''} S''_i,\\ U \cup P^- &= \{1, \ldots, p\}.\end{aligned}$$

Let $\mathscr{E}_1 \colon \check{z}_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C, (w_j : F_j)_{j\in T^-}, (v_j : G_j)_{j\in V} \Rightarrow \widetilde{C}_1^{\widetilde{k}} \to C$ be the following normal deduction:

(104) $\mathscr{E}_1 =$

$$\dfrac{\dfrac{\dfrac{\check{z}_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C \qquad \begin{matrix}(w_j : F_j)_{j\in T^-}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'}\\ \hat{\mathscr{C}}_1^k\\ C_1^k\end{matrix}}{((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C} \to E \qquad \left(\dfrac{\begin{matrix}(v_j : G_j)_{j\in P_i^+\cup V_i}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i}\\ \hat{\mathscr{H}}_i\\ H_i\end{matrix}}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+}\right)_{i=1}^q}{C} \to E}{\widetilde{C}_1^{\widetilde{k}} \to C} \to I, \widetilde{u}_1^{\widetilde{k}}$$

where $\hat{\mathscr{C}}_1^k$ is as in (49). We show

(105) $|\hat{\mathscr{D}}_1|_\beta, (\mathscr{D}'_j)_{j\in T^-}, (\mathscr{B}_j)_{j\in V}$ is an $\varnothing$-interpolant to $\widetilde{\mathscr{D}}_1$ via $\mathscr{E}_1$.

That $\mathscr{E}_1$ satisfies condition (I4) of Definition 16 can be checked using (F) and (K). It remains to show $\mathscr{E}_1[\hat{\mathscr{D}}_1/\check{z}_1, (\mathscr{D}'_j/w_j)_{j\in T^-}, (\mathscr{B}_j/v_j)_{j\in V}] \twoheadrightarrow_\beta \widetilde{\mathscr{D}}_1$.

$$\mathscr{E}_1[\hat{\mathscr{D}}_1/\check{z}_1, (\mathscr{D}'_j/w_j)_{j\in T^-}, (\mathscr{B}_j/v_j)_{j\in V}]$$

$=$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'_1 \cup \Gamma''_1 \cup \bigcup_{i\in P^+} \Gamma''_i \\ \hat{\mathscr{D}}_1 \\ C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C\end{array} \qquad \begin{array}{c}\bigcup_{j\in T^-} \Gamma'_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \hat{\mathscr{C}}_1^k[(\mathscr{D}'_j/w_j)_{j\in T^-}] \\ C_1^k\end{array}}{((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C} \to E \qquad \left(\begin{array}{c}(v_j : G_j)_{j\in P_i^+}, \bigcup_{j\in V_i} \Gamma''_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i} \\ \hat{\mathscr{H}}_i[(\mathscr{B}_j/v_j)_{j\in V_i}] \\ \dfrac{H_i}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+}\end{array}\right)_{i=1}^q}{C} \to E}{}}{\widetilde{C}_1^k \to C} \to I, \widetilde{u}_1^k$$

$\twoheadrightarrow_\beta$ (by (13))

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'_1 \\ \mathscr{D}'_1 \\ C_1^k \to (A_1^l \to B) \to C\end{array} \quad \begin{array}{c}\bigcup_{j\in T^-} \Gamma'_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \hat{\mathscr{C}}_1^k[(\mathscr{D}'_j/w_j)_{j\in T^-}] \\ C_1^k\end{array}}{(A_1^l \to B) \to C} \to E \qquad \dfrac{\dfrac{\begin{array}{c}\Gamma''_1, ((u_j : A_j)^\circ)_{j=1}^l \\ \mathscr{B}_1 \\ H_1^q \to B\end{array} \quad \left(\dfrac{\begin{array}{c}(v_j : G_j)_{j\in P_i^+}, \bigcup_{j\in V_i} \Gamma''_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i} \\ \hat{\mathscr{H}}_i[(\mathscr{B}_j/v_j)_{j\in V_i}] \\ \dfrac{H_i}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+}\end{array} \quad \left(\begin{array}{c}\Gamma''_j, ((u_h : A_h)^\circ)_{h=1}^l \\ \mathscr{B}_j \\ G_j\end{array}\right)_{j\in P_i^+}}{H_i} \to E\right)_{i=1}^q}{B} \to E}{A_1^l \to B} \to I, u_1^l}{C} \to E}{\widetilde{C}_1^k \to C} \to I, \widetilde{u}_1^k$$

$\twoheadrightarrow_\beta$ (by (53))

$$\dfrac{\dfrac{\begin{array}{c}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A_1^l \to B) \to C\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Gamma''_1, ((u_j : A_j)^\circ)_{j=1}^l \\ \mathscr{B}_1 \\ H_1^q \to B\end{array} \quad \left(\dfrac{\begin{array}{c}(v_j : G_j)_{j\in P_i^+}, \bigcup_{j\in V_i} \Gamma''_j, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i} \\ \hat{\mathscr{H}}_i[(\mathscr{B}_j/v_j)_{j\in V_i}] \\ \dfrac{H_i}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+}\end{array} \quad \left(\begin{array}{c}\Gamma''_j, ((u_h : A_h)^\circ)_{h=1}^l \\ \mathscr{B}_j \\ G_j\end{array}\right)_{j\in P_i^+}}{H_i} \to E\right)_{i=1}^q}{B} \to E}{A_1^l \to B} \to I, u_1^l}{C} \to E}{\widetilde{C}_1^k \to C} \to I, \widetilde{u}_1^k$$

$\twoheadrightarrow_\beta$

$$\dfrac{\dfrac{\begin{array}{c}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A_1^l \to B) \to C\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Gamma''_1, ((u_j : A_j)^\circ)_{j=1}^l \\ \mathscr{B}_1 \\ H_1^q \to B\end{array} \quad \left(\begin{array}{c}\bigcup_{j\in V_i\cup P_i^+} \Gamma''_j, ((u_j : A_j)^\circ)_{j=1}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K''_i} \\ \hat{\mathscr{H}}_i[(\mathscr{B}_j/v_j)_{j\in V_i\cup P_i^+}] \\ H_i\end{array}\right)_{i=1}^q}{B} \to E}{A_1^l \to B} \to I, u_1^l}{C} \to E}{\widetilde{C}_1^k \to C} \to I, \widetilde{u}_1^k$$

$\twoheadrightarrow_\beta$ (by (102) and (101))

$$
\dfrac{\begin{array}{c}\widetilde{\Gamma}'_1, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K'} \\ \widetilde{\mathscr{D}}'_1 \\ (A_1^l \to B) \to C\end{array} \qquad \dfrac{\begin{array}{c}\widetilde{\Gamma}''_1, (\widetilde{u}_i : \widetilde{C}_i)_{i\in K''}, ((u_j : A_j)^\circ)_{j=1}^l \\ \widetilde{\mathscr{D}}''^{-}_1 \\ B\end{array}}{A_1^l \to B}\ {\to}I, u_1^l}{\dfrac{C}{\widetilde{C}_1^k \to C}\ {\to}I, \widetilde{u}_1^k}\ {\to}E
$$

= (by (41) and (93))

$$\widetilde{\mathscr{D}}_1$$

The output $\mathscr{D}_1^m, \mathscr{D}_0$ of the new method is the result of applying the pruning procedure to $(\check{\mathscr{D}}_i \colon \check{\Gamma}_i \Rightarrow E_i)_{i=1}^{\check{m}}, \check{\mathscr{D}}_0 \colon (\check{z}_i : E_i)_{i=1}^{\check{m}}, \Delta \Rightarrow C$, where

$$
\begin{aligned}
\check{\mathscr{D}}_1^m &= |\hat{\mathscr{D}}_1|_\beta, (\mathscr{D}'_i)_{i\in N}, (\mathscr{B}_i)_{i\in P^-}, \\
(\check{z}_i : E_i)_{i=1}^{\check{m}} &= \check{z}_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C, (w_i : F_i)_{i\in N}, (v_i : G_i)_{i\in P^-},
\end{aligned}
$$

as described in Case 2.2.2 of the new method. Let $n_1, \ldots, n_r$ and $p_1, \ldots, p_s$ list the elements of N and P^-, respectively, in increasing order, so that $\check{m} = 1 + r + s$. Let $\mu(i)$ and $\mathscr{M}_i$ be as in the description of prune$(\check{\mathscr{D}}_1^{\check{m}}, \check{\mathscr{D}}_0)$. We have $\mu(1) = 1$ and $\mathscr{D}_1 = \check{\mathscr{D}}_1 = |\hat{\mathscr{D}}_1|_\beta$ is an $\varnothing$-interpolant to itself via $\mathscr{M}_1$. For $i = 1, \ldots, r$, $\mathscr{D}_{\mu(1+i)}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_{1+i} = \mathscr{D}'_{n_i}$ via $\mathscr{M}_{1+i}$. For $i = 1, \ldots, s$, $\mathscr{D}_{\mu(1+r+i)}$ is an $\varnothing$-interpolant to $\check{\mathscr{D}}_{1+r+i} = \mathscr{B}_{p_i}$ via $\mathscr{M}_{1+r+i}$. We define subsets $S_1^{\widetilde{m}}$ of $\{1, \ldots, m\}$ and deductions $\mathscr{E}_i \colon (z_j : E_j)_{j\in S_i} \Rightarrow \widetilde{E}_i)_{i=1}^{\widetilde{m}}$ as follows:

$$
S_i = \begin{cases} \{1\} \cup \{\, \mu(1+j) \mid n_j \in T^- \,\} \cup \{\, \mu(1+r+j) \mid p_j \in V \,\} & \text{if } i = 1, \\ \{\, \mu(1+j) \mid n_j \in S'_i \,\} & \text{if } i \in M' - \{1\}, \\ \{\, \mu(1+r+j) \mid p_j \in S''_i \,\} & \text{if } i \in M'' - M' - \{1\}. \end{cases}
$$

$$
\mathscr{E}_i = \begin{cases} |\hat{\mathscr{E}}_1[\mathscr{M}_1/\check{z}_1, (\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V}]|_\beta & \text{if } i = 1, \\ |\mathscr{E}'_i[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in S'_i}]|_\beta & \text{if } i \in M' - \{1\}, \\ |\mathscr{E}''_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in S''_i}]|_\beta & \text{if } i \in M'' - M' - \{1\}. \end{cases}
$$

($\hat{\mathscr{E}}_1, \mathscr{E}'_i, \mathscr{E}''_i$ are given in (104), (48), and (98), respectively.) We show that $S_1^{\widetilde{m}}$ and $\mathscr{E}_1^{\widetilde{m}}$ satisfy conditions 2–4.

Condition 2 follows from (105), (48.ii.b), and (98.ii.b), using the property of $\mathscr{M}_i$ mentioned above. Condition 4 easily follows from (23).

It remains to prove condition 3. From (35) and (104), we see that

$$
\widetilde{\mathscr{D}}_0[(\mathscr{E}_i/\widetilde{z}_i)_{i=1}^{\widetilde{m}}] =_\beta \widetilde{\mathscr{D}}_0[\hat{\mathscr{E}}_1[\mathscr{M}_1/\check{z}_1, (\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V}]/\widetilde{z}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i=2}^{\widetilde{m}}]
$$

$$
\twoheadrightarrow_\beta \quad \dfrac{\dfrac{\begin{array}{c} z_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C \\ \mathscr{M}_1 \\ C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C \end{array} \qquad \widetilde{\mathscr{L}}_1^k}{((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C}\ {\to}E \qquad \widetilde{\mathscr{R}}_1^q}{C}\ {\to}E
$$

where

$$
\begin{array}{c} (z_h : E_h)_{h\in\{\,\mu(1+j) \mid n_j\in T^-\,\}\cup\bigcup_{i\in M'} S_i}, \Delta' \\ \widetilde{\mathscr{L}}_1^k = \quad \hat{\mathscr{C}}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}] \\ C_1^k \end{array}
$$

$$\widetilde{\mathscr{R}}_i = \dfrac{\dfrac{\begin{array}{c}(v_j : G_j)_{j\in P_i^+}, (z_g : E_g)_{g\in\{\mu(1+r+j) \mid p_j\in V_i\}\cup\bigcup_{h\in M_j} S_h}, \bigcup_{j\in K_i''} \widetilde{\Delta}_j \\ \hat{\mathscr{H}}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}]\end{array}}{H_i}}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+} \qquad \text{for } i = 1, \ldots, q.$$

Since $\mathscr{D}_0 = |\check{\mathscr{D}}_0[(\mathscr{M}_i/\check{z}_i)_{i=1}^{\check{m}}]|_\beta$, where $\check{\mathscr{D}}_0$ is given in (14), $\mathscr{D}_0$ is the normal form of

$$\dfrac{\dfrac{\dfrac{z_1 : C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C}{C_1^k \to ((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C}\mathscr{M}_1 \qquad \mathscr{L}_1^k}{((G_j)_{j\in P_i^+} \to H_i)_{i=1}^q \to C} \to E \qquad \mathscr{R}_1^q}{C} \to E$$

where

$$\mathscr{L}_1^k = \begin{array}{c}(z_{\mu(1+i)} : E_{\mu(1+i)})_{n_i\in N}, \Delta' \\ \mathscr{C}_1^k[(\mathscr{M}_{1+i}/w_{n_i})_{n_i\in N}] \\ C_1^k\end{array}$$

$$\mathscr{R}_i = \dfrac{\dfrac{\begin{array}{c}(v_j : G_j)_{j\in P_i^+}, (z_{\mu(1+r+j)} : E_{\mu(1+r+j)})_{p_j\in P_i^-}, \Delta_i'' \\ \mathscr{H}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P_i^-}]\end{array}}{H_i}}{(G_j)_{j\in P_i^+} \to H_i} \to I, (v_j)_{j\in P_i^+} \qquad \text{for } i = 1, \ldots, q.$$

So it suffices to show

(106) $\quad \hat{\mathscr{C}}_1^k[(\mathscr{M}_{1+j}/w_{n_j})_{n_j\in T^-}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_i/\widetilde{z}_i)_{i\in M_j}]/\widetilde{u}_j)_{j\in K'}] =_\beta \mathscr{C}_1^k[(\mathscr{M}_{1+i}/w_{n_i})_{n_i\in N}],$

and

(107)
$$\hat{\mathscr{H}}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}] =_\beta \mathscr{H}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P_i^-}]$$

for $i = 1, \ldots, q$.

We can prove (106) in exactly the same way as (80) of Case 2.2b.1.2.

We prove (107). By (97), (98.ii.a), and condition 2, we get

(108) $(\mathscr{B}_j)_{j\in W}, (\mathscr{D}_j)_{j\in\bigcup_{i\in M''} S_i}$ is an interpolant to $\mathscr{B}$ with respect to $(\Gamma'', ((u_j : A_j)^\circ)_{j=1}^l; \Delta'')$ via the normal form of $\widetilde{\mathscr{B}}_0[\mathscr{G}/\widetilde{v}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}]$: $(v_j : G_j)_{j\in W}, (z_j : E_j)_{j\in\bigcup_{i\in M''} S_i}, \Delta'' \Rightarrow B$.

Applying the induction hypothesis again to $\mathscr{B}$ with respect to (108) and noting Lemma 32, we obtain elements $(\hat{\tau}(j))_{j\in\bigcup_{i\in M''} S_i}$ of $\{1, \ldots, p\}$ and deductions $(\mathscr{I}_j : v_j : G_j \Rightarrow G_j)_{j\in W}, (\mathscr{T}_j : v_{\hat{\tau}(j)} : G_{\hat{\tau}(j)} \Rightarrow E_j)_{j\in\bigcup_{i\in M''} S_i}$ such that

(109) i. $W \cup \{\, \hat{\tau}(j) \mid j \in \bigcup_{i\in M''} S_i \,\} = \{1, \ldots, p\}$;

ii. a. $\mathscr{B}_j$ is an $\varnothing$-interpolant to itself via $\mathscr{I}_j$ for each $j \in W$;

b. $\mathscr{B}_{\hat{\tau}(j)}$ is an $\varnothing$-interpolant to $\mathscr{D}_j$ via $\mathscr{T}_j$ for each $j \in \bigcup_{i\in M''} S_i$;

iii. $\widetilde{\mathscr{B}}_0[\mathscr{G}/\widetilde{v}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}][(\mathscr{I}_j/v_j)_{j\in W}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}] \twoheadrightarrow_\beta \mathscr{B}_0$.

By (98.iv.a),

$$\mathscr{G}[(\mathscr{I}_j/v_j)_{j\in W}] \twoheadrightarrow_\beta \mathscr{G}. \tag{110}$$

We have

$$\begin{aligned}
&\widetilde{\mathscr{B}}_0[\mathscr{G}/\widetilde{v}_1, (\mathscr{E}_i/\widetilde{z}_i)_{i\in M''}][(\mathscr{I}_j/v_j)_{j\in W}, (\mathscr{T}_j/z_j)_{j\in\bigcup_{i\in M''} S_i}] \\
&= \widetilde{\mathscr{B}}_0[\mathscr{G}[(\mathscr{I}_j/v_j)_{j\in W}]/\widetilde{v}_1, (\mathscr{E}_i[(\mathscr{T}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}] \\
&\twoheadrightarrow_\beta \widetilde{\mathscr{B}}_0[\mathscr{G}/\widetilde{v}_1, (\mathscr{E}_i[(\mathscr{T}_j/z_j)_{j\in S_i}]/\widetilde{z}_i)_{i\in M''}] \quad \text{by (110)} \\
&= \dfrac{\dfrac{v_1 : H_1^q \to B \quad \begin{pmatrix} (v_j : G_j)_{j\in W_i}, (\widetilde{u}_j : \widetilde{C}_j)_{j\in K_i''} \\ \hat{\mathscr{H}}_i \\ H_i \end{pmatrix}_{i=1}^q}{B} \to E \qquad \dfrac{\dfrac{B}{(\widetilde{C}_j)_{j\in K''} \to B} \to I, (\widetilde{u}_j)_{j\in K''} \quad \begin{pmatrix} (v_{\hat{\tau}(g)} : G_{\hat{\tau}(g)})_{g\in\bigcup_{h\in M_j} S_h}, \widetilde{\Delta}_j \\ \widetilde{\mathscr{C}}_j[(\mathscr{E}_h[(\mathscr{T}_g/z_g)_{g\in S_h}]/\widetilde{z}_h)_{h\in M_j}] \\ \widetilde{C}_j \end{pmatrix}_{j\in K''}}{B} \to E}{} \\
&\qquad \text{by (96) and (99)} \\
&\twoheadrightarrow_\beta \dfrac{v_1 : H_1^q \to B \quad \begin{pmatrix} (v_f : G_f)_{f\in W_i\cup\{\,\hat{\tau}(g) \mid g\in\bigcup_{j\in K_i''}\bigcup_{h\in M_j} S_h\,\}}, \bigcup_{j\in K_i''} \widetilde{\Delta}_j \\ \hat{\mathscr{H}}_i[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_h[(\mathscr{T}_g/z_g)_{g\in S_h}]/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}] \\ H_i \end{pmatrix}_{i=1}^q}{B} \to E
\end{aligned}$$

Therefore, by (12) and (109.iii),

$$\hat{\mathscr{H}}_i[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_h[(\mathscr{T}_g/z_g)_{g\in S_h}]/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}] \twoheadrightarrow_\beta \mathscr{H}_i, \tag{111}$$

which implies that

$$V_i \cup \{\, \hat{\tau}(g) \mid g \in \bigcup_{j\in K_i''} \bigcup_{h\in M_j} S_h \,\} = P_i^-. \tag{112}$$

Let $(\tau(j))_{j\in\bigcup_{i\in M''} S_i}$ be such that $\hat{\tau}(j) = p_{\tau(j)}$ for each $j \in \bigcup_{i\in M''} S_i$. By (112),

$$V_i \cup \{\, p_{\tau(g)} \mid g \in \bigcup_{j\in K_i''} \bigcup_{h\in M_j} S_h \,\} = P_i^-. \tag{113}$$

By (25), (109.ii.b) implies that for $j \in \bigcup_{i\in M''} S_i$, $\mu(1 + r + \tau(j)) = j$ and $\mathscr{D}_j$ is an $\varnothing$-interpolant to $\mathscr{B}_{p_{\tau(j)}}$ via $\mathscr{M}_{1+r+\tau(j)} \colon z_j : E_j \Rightarrow G_{p_{\tau(j)}}$. It follows that $\mathscr{D}_j$ is an $\varnothing$-interpolant to itself via the normal form of $\mathscr{T}_j[\mathscr{M}_{1+r+\tau(j)}/v_{p_{\tau(j)}}] \colon z_j : E_j \Rightarrow E_j$. Hence by condition 4,

$$\mathscr{E}_i[(\mathscr{T}_j[\mathscr{M}_{1+r+\tau(j)}/v_{p_{\tau(j)}}]/z_j)_{j\in S_i}] \twoheadrightarrow_\beta \mathscr{E}_i \quad \text{for } i \in M''. \tag{114}$$

Now we can show (107) as follows:

$$\begin{aligned}
&\hat{\mathscr{H}}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_h/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}] \\
&=_\beta \hat{\mathscr{H}}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in V_i}, (\widetilde{\mathscr{C}}_j[(\mathscr{E}_h[(\mathscr{T}_g[\mathscr{M}_{1+r+\tau(g)}/v_{p_{\tau(g)}}]/z_g)_{g\in S_h}]/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}] \\
&\qquad \text{by (44), (100), and (114)} \\
&= \hat{\mathscr{H}}_i[(\widetilde{\mathscr{C}}_j[(\mathscr{E}_h[(\mathscr{T}_g/z_g)_{g\in S_h}]/\widetilde{z}_h)_{h\in M_j}]/\widetilde{u}_j)_{j\in K_i''}][(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P_i^-}] \quad \text{by (113)} \\
&\twoheadrightarrow_\beta \mathscr{H}_i[(\mathscr{M}_{1+r+j}/v_{p_j})_{p_j\in P_i^-}] \quad \text{by (111)}.
\end{aligned}$$

This concludes the proof of Claim C. □

Remark. Using the algorithm given in the proof of Claim B, it is not hard to see that the function prune can be computed in polynomial time. Since the complexity of the new method is clearly dominated by the complexity of prune, it follows that the new method itself can be implemented to run in polynomial time.

Example 46. Consider the following normal deduction $\mathscr{D}$: $x_1 : ((p_1 \to p_2) \to p_5) \to p_6, y_1 : p_4 \to p_5, x_2 : p_3 \to p_4, y_2 : p_2 \to p_3, x_3 : p_1 \Rightarrow p_6$ from Example 31:

$$
\begin{prooftree}
\AxiomC{$x_1 : ((p_1 \to p_2) \to p_5) \to p_6$}
\AxiomC{$y_1 : p_4 \to p_5$}
\AxiomC{$x_2 : p_3 \to p_4$}
\AxiomC{$y_2 : p_2 \to p_3$}
\AxiomC{$u : p_1 \to p_2$}
\AxiomC{$x_3 : p_1$}
\RightLabel{$\to E$}
\BinaryInfC{p_2}
\RightLabel{$\to E$}
\BinaryInfC{p_3}
\RightLabel{$\to E$}
\BinaryInfC{p_4}
\RightLabel{$\to E$}
\BinaryInfC{p_5}
\RightLabel{$\to I, u$}
\UnaryInfC{$(p_1 \to p_2) \to p_5$}
\RightLabel{$\to E$}
\BinaryInfC{p_6}
\end{prooftree}
$$

Let us see how the new method works on this deduction with respect to the partition

$$(x_1 : ((p_1 \to p_2) \to p_5) \to p_6,\ x_2 : p_3 \to p_4,\ x_3 : p_1\ ;\ y_1 : p_4 \to p_5,\ y_2 : p_2 \to p_3).$$

Let $\mathscr{D}^{(i)}$ the subdeduction whose endformula is p_i. We list the λ-terms (along with their type) corresponding to the interpolants computed by the new method when given $\mathscr{D}^{(i)}$ (together with the relevant partition) as input:

$\mathscr{D}^{(2)}$ by Case 2.2.2.	$ux_3 : p_2$
$\mathscr{D}^{(3)}$ by Case 2.1.	$ux_3 : p_2$
$\mathscr{D}^{(4)}$ by Case 2.2.1.1.	$x_2 : p_3 \to p_4,\ ux_3 : p_2$
$\mathscr{D}^{(5)}$ by Case 2.1.	$x_2 : p_3 \to p_4,\ ux_3 : p_2$
$\mathscr{D}^{(6)}$ by Case 2.2.1.2.	$\lambda v.x_1(\lambda u.v(ux_3)) : (p_2 \to p_5) \to p_6,\ x_2 : p_3 \to p_4$

The output of the new method on $\mathscr{D}$ is the sequence $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_0$ of Example 31. (In this example, the pruning procedure does not affect the outcome.)

Remark. We note that Theorem 45 relativizes to *substructural logics* (*BCI-logic*, *BCK-logic*, $R_{\to}$). Condition (I1) of Definition 16 is strengthened to "$\Gamma_1, \ldots, \Gamma_m = \Gamma$" in the case of *BCI*- and *BCK*-logic. These logics do not require the pruning procedure, and the proof of Theorem 45 is accordingly greatly simplified.

Remark. We may choose to treat deductions modulo η-equality, as is often done in typed λ-calculus. This will simplify the proof of Theorem 45 in many places. In particular, Cases 2.2.1.1 and 2.2.1.2 of the Induction Step of the new method will no longer need to be distinguished. Of course, the resulting statement of the theorem will become weaker.

4 Discussion

We have presented a new algorithm for computing an interpolant to a given normal natural deduction $\mathscr{D}$ (with respect to a partition of its set of assumptions) in the implicational fragment of intuitionistic logic. From among many interpolants to $\mathscr{D}$, this algorithm picks out a strongest one in a certain natural sense, but our notion of an "interpolant", given by Definition 16, is somewhat restricted because not all interpolation sequences (in the sense of section 1) can be obtained from interpolants.

For instance, consider the following deduction:

$$\mathscr{D} = \dfrac{\dfrac{y_1 : r \to r \to s \qquad \dfrac{x : (p \to q) \to r \quad y_2 : p \to q}{r}\,{\to}E}{r \to s}\,{\to}E \qquad \dfrac{x : (p \to q) \to r \quad \dfrac{y_3 : q}{p \to q}\,{\to}I}{r}\,{\to}E}{s}\,{\to}E$$

The one-formula sequence $(p \to q) \to r$ is an interpolation sequence to $(p \to q) \to r, r{\to}r{\to}s, p{\to}q, q \Rightarrow s$ with respect to the partition $((p{\to}q){\to}r; r{\to}r{\to}s, p{\to}q, q)$. The associated deductions

$$\mathscr{D}_1 = x : (p \to q) \to r$$
$$\mathscr{D}_0 = \mathscr{D}[z_1 : (p \to q) \to r/x]$$

satisfy conditions (I1), (I2), and (I3) of Definition 16, but not (I4), so $\mathscr{D}_1$ does not count as an interpolant. Up to $\beta\eta$-equality, there is only one interpolant to $\mathscr{D}$ (with respect to the partition in question), namely $\mathscr{D}_1, \mathscr{D}_2$, where

$$\mathscr{D}_2 = \dfrac{\dfrac{x : (p \to q) \to r \quad \dfrac{v : q}{p \to q}\,{\to}I}{r}\,{\to}E}{q \to r}\,{\to}I, v$$

This interpolant gives an interpolation sequence $(p \to q) \to r, q \to r$ which is more complex than the above interpolation sequence $(p{\to}q){\to}r$. A weaker definition of an interpolant is conceivable under which (the sequence consisting of) $\mathscr{D}_1$ counts as an 'interpolant' to $\mathscr{D}$, but interpolants in such a weaker sense cannot be constructed inductively. (Note that $x{:}(p{\to}q){\to}r$ is not an 'interpolant' in any reasonable sense to the immediate subdeduction of $\mathscr{D}$ whose endformula is r.) Our definition of an interpolant (Definition 16) is the one that is naturally extracted from the existing syntactical methods for proving interpolation.

It may also be worth mentioning that the interpolation sequence associated with a strongest interpolant may not be one of the simplest ones among all the interpolation sequences obtained from interpolants. For instance,

$$\mathscr{D} = \dfrac{\dfrac{x : p \to p \to q \quad y : p}{p \to q}\,{\to}E \qquad y : p}{q}\,{\to}E$$

has an interpolant

$$\dfrac{\dfrac{\dfrac{x : p \to p \to q \quad u : p}{p \to q}\,{\to}E \qquad u : p}{q}\,{\to}E}{p \to q}\,{\to}I, u$$

which is strictly less strong than the strongest interpolant:

$$x : p \to p \to q.$$

The main result of this paper should be compared to a result in Pitts 1992, which states that the set of interpolation formulas to a given sequent $\Gamma, \Delta \Rightarrow C$ in intuitionistic propositional logic has a least and a greatest element with respect to the usual preorder given by

$$A \leq B \quad \text{iff} \quad \vdash A \Rightarrow B.$$

This result by Pitts is different from our main result in a number of respects. Firstly, Pitts' proof of his result does not take into account 'intensional' properties of interpolants as expressed in our condition (I2) of Definition 16 or condition 3 of Definition 5.[17] Secondly, Pitts' result essentially depends on the presence of conjunction and disjunction and it does not specialize to the implicational fragment of intuitionistic logic. Thirdly, his result makes essential use of Weakening and does not relativize to substructural logics. Looking from the opposite angle, since not all interpolation sequences are obtained from interpolants in the sense of Definition 16, our main result does not imply that Pitts' result holds of the implicational fragment of intuitionistic logic.

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[17]Indeed, Pitts' proof uses a system of intuitionistic propositional logic that is complete only in the 'extensional' sense of being able to derive all intuitionistic validities. The system has a finite proof search space and not all normal λ-terms correspond to proofs of implicational formulas in this system.

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