Dynamic Generalized Quantifiers and Monotonicity*

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1 Introduction

It has long been recognized that certain types of anaphoric dependencies in natural language cannot be straightforwardly captured by variable binding in standard logic. They are 'intersentential' and donkey anaphora, which have been widely discussed in the literature since Geach 1962. In the last decade, they motivated various similar proposals of so-called 'discourse semantics', including Kamp 1981, Heim 1982, Barwise 1987, Rooth 1987, and Schubert and Pelletier 1989. Groenendijk and Stokhof's recent (1991) dynamic predicate logic (DPL) is an attempt to capture the basic insights of earlier theories in the form of a minimal modification of first-order logic.

Its similarity with first-order logic gives the formalism of DPL some distinct advantages. DPL is easy to use; for those familiar with first-order logic, it is easy to develop intuitions about DPL. As a logical system closely related to first-order logic, DPL naturally brings standard logical concerns into the picture, like the notion of logical consequence and its syntactic characterization. It is also easy to consider various extensions of DPL analogous to well-known extensions of first-order logic.

This paper concerns the system of DPL augmented with generalized quantifiers. DPL with generalized quantifiers can be seen as a refinement of first-order logic with generalized quantifiers, just as DPL is a refinement of first-order logic. Systems of first-order logic with generalized quantifiers have

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been studied mainly in mathematical logic, but when the system has a two-place generalized quantifier symbol for each natural language determiner, it can be considered as a fairly good model of a certain extensional fragment of natural language. Such a fragment is studied in generalized quantifier theory (van Benthem 1986, Westerståhl 1989), and DPL with generalized quantifiers is expected to add to the theory the capacity to treat anaphoric subtlety of natural language.

The kind of generalized quantifier that is of interest here is what Groenendijk and Stokhof would call *internally dynamic* generalized quantifiers. They allow an indirect binding relation to obtain between dynamic existential quantifiers in their first argument and corresponding variables in their second argument. An empirical motivation for studying such generalized quantifiers comes from donkey sentences with relative clauses. The correspondence is shown in (1):

(1) a. Det $\overline{\mathrm{N}}$ $[_{\overline{\mathrm{S}}} \ldots [_{\mathrm{NP}_i}$ a $\overline{\mathrm{N}}] \ldots]$ $[_{\mathrm{VP}} \ldots [_{\mathrm{NP}_i}$ pronoun$] \ldots]$

b. $\mathscr{Q}x(\ldots \mathscr{E}y \ldots, \ldots y \ldots)$

(1a) is a donkey sentence with a relative clause, like *Every farmer who owns a donkey beats it*. (1b) is a formula of dynamic predicate logic with generalized quantifiers. $\mathscr{Q}$ is an internally dynamic generalized quantifier, and $\mathscr{E}$ is the dynamic existential quantifier of DPL. The anaphoric relation in (1a) is modelled by the indirect binding relation between $\mathscr{E}y$ and y mediated by $\mathscr{Q}$ in (1b).

An interesting task that suggests itself is to develop a theory of dynamic generalized quantifiers, something analogous to the well-established theory of ordinary generalized quantifiers (van Benthem 1986, Westerståhl 1989). Such a theory should formulate dynamic notions analogous to static ones found in generalized quantifier theory and prove formal results about dynamic generalized quantifiers. This paper can do no more than suggest how such a theory might begin. While the standard generalized quantifier theory is given at the denotational level, it will be convenient, at least for a start, to speak of dynamic generalized quantifiers mainly in syntactic terms, using analogy with first-order logic with generalized quantifiers.

Special attention is paid to a notion of *Monotonicity* suitable for internally dynamic generalized quantifiers. This is in part inspired by a more general problem of accounting for *monotonicity inference* in dynamic contexts. To illustrate the problem, the following non-inference shows that one must take care in drawing monotonicity inference in the presence of donkey anaphora:[1]

(2)
$$\frac{\begin{array}{c}[\![\textit{man who owns a garden}]\!] \subseteq [\![\textit{man who owns a house}]\!] \\ \text{No man who owns a house sprinkles it}\end{array}}{\text{No man who owns a garden sprinkles it}}$$

[1] The example is adapted from van Benthem 1987.

Here, the first premise is that the set of men who own a garden is a subset of the set of men who own a house, or every man who owns a garden is a man who owns a house. From the fact that the determiner *no* is downward monotone in the first argument,[2] one might expect that the replacement of *man who owns a house* in the second premise by *man who owns a garden* is truth-preserving, which in fact it is not. The problem is obviously caused by the presence of a donkey pronoun *it*; compare the validity of the inference from *No man who owns a house is poor* to *No man who owns a garden is poor*, under the same assumption. Nevertheless, it is not the case that monotonicity inference does not make sense in donkey sentences. Both of the following are valid instances of monotonicity inference:

(3) $$\frac{\text{No farmer who owns a donkey beats it}}{\text{No farmer who owns and feeds a donkey beats it}}$$

(4) $$\frac{\text{No farmer who owns a donkey beats it}}{\text{No farmer who owns a female donkey beats it}}$$

What is called for is an appropriate dynamic sense of monotonicity more restrictive than the usual one which accounts for the invalidity of (2) and the validity of (3) and (4).

Our concern for Monotonicity for dynamic generalized quantifiers has another empirical motivation which is related to the issue of monotonicity inference. Here, the task is to predict the interpretation of donkey sentences with relative clauses from the monotonicity properties of the determiner. It has been observed that two distinct types of interpretations are found in donkey sentences with determiners and relative clauses. One interpretation, called the *strong reading*, allows a paraphrase with universal quantification over donkeys, and the other interpretation, called the *weak reading*, allows a paraphrase with existential quantification over donkeys.[3] The interpretation standardly associated with *Every farmer who owns a donkey beats it* is the strong reading: *Every farmer who owns a donkey beats every donkey he owns*. The (only) interpretation of *No farmer who owns a donkey beats it* is

[2] A determiner is said to be downward monotone in the first-argument (↓MON) if its denotation Q_M in any universe M satisfies the following:

$$\text{for all } A, A', B \subseteq M,\ Q_M AB \text{ and } A' \subseteq A \text{ imply } Q_M A'B.$$

Replacing $A' \subseteq A$ in the above definition by $A \subseteq A'$ gives the definition of upward monotonicity in the first argument (↑MON). Monotonicity in the second argument (MON↑, MON↓) is defined analogously. Monotonicity in the first (second) argument is also called left (right) monotonicity.

[3] Strong and weak readings of donkey sentences have been discussed by Rooth (1987), Chierchia (1990, 1992), and Gawron, Nerbonne, and Peters (1991), among others. The terms *strong reading* and *weak reading* are apparently due to Chierchia (1990).

the weak reading: *No farmer who owns a donkey beats a donkey he owns*. The interesting fact is that in many cases, one or the other interpretation is the only available one, or at least strongly preferred, and which reading is available correlates with the monotonicity properties of the determiner. Compare the following sentences, of which the available reading and the monotonicity properties of the determiner are indicated.

(5) Every student who borrowed a book from Peter returned it
= Every student who borrowed a book from Peter returned every book he or she borrowed from Peter (strong reading, $\downarrow$MON$\uparrow$)

(6) No student who borrowed a book from Peter returned it
= No student who borrowed a book from Peter returned a book he or she borrowed from Peter (weak reading, $\downarrow$MON$\downarrow$)

(7) At least two students who borrowed a book from Peter returned it
= At least two students who borrowed a book from Peter returned a book they borrowed from Peter (weak reading, $\uparrow$MON$\uparrow$)

(8) Not every student who borrowed a book from Peter returned it
= Not every student who borrowed a book from Peter returned every book he or she borrowed from Peter (?) (strong reading, $\uparrow$MON$\downarrow$)

We find the correlation between monotonicity properties of determiners and interpretations of donkey sentences given in Table 1. The data is actually quite

	Available reading(s)	Determiners
$\uparrow$MON$\uparrow$	Weak reading only	*a, some, several, at least n, many*
$\uparrow$MON$\downarrow$	Strong reading preferred?	*not every, not all*
$\downarrow$MON$\uparrow$	Strong reading preferred	*every, all, FC any*
$\downarrow$MON$\downarrow$	Weak reading only	*no, few, at most n*
$\not\uparrow\not\downarrow$MON$\uparrow$	Both	*most*

Table 1: Monotonicity of determiners and interpretations of donkey sentences.

complex, and we cannot elaborate on Table 1 here.[4] In this paper, we will simply assume the data as summarized in Table 1, and give an explanation of the observed correlation using the notion of Monotonicity for dynamic generalized quantifiers. The key fact is that the selected reading of a donkey sentence with a left monotone determiner is the one on which monotonicity inference like (3) and (4) comes out valid.

The paper is organized as follows. In Section 2, we briefly look at first-order logic with generalized quantifiers from a natural language perspective.

[4]For detailed discussions of weak and strong readings of donkey sentences with relative clauses and of monotonicity inference, the reader is referred to Kanazawa 1993.

In Section 3, we introduce our version of dynamic predicate logic. Section 4 is devoted to the system of DPL augmented with static and dynamic generalized quantifiers. In Section 4.1, we see two ways of defining dynamic generalized quantifiers in terms of static ones and dynamic connectives of DPL. In Section 4.2, we consider dynamic notions of Conservativity. In Section 4.3, a suitable dynamic notion of Monotonicity is formulated, which is then used to explain the correlation given in Table 1. In Section 4.4, we demonstrate that, under minimal assumptions, dynamic double monotonicity actually serves as an implicit definition of a dynamic generalized quantifier in terms of a static one. In Section 4.5, a correlation between left monotonicity and model-theoretic preservation properties is extended to the dynamic case. Proof of the results in Sections 4.4 and 4.5 is relegated to Section 4.6.

2 First-Order Logic with Generalized Quantifiers

Before turning to dynamic predicate logic, let us briefly look at ordinary first-order logic with generalized quantifiers. The purpose of this section is to express various general and special properties of quantifiers as formulas (in the case of local conditions on the denotation in each model) or model-theoretic properties of formulas (in the case of global conditions operating across models) in the language of first-order logic with generalized quantifiers. This will prove convenient when recasting these properties in the dynamic setting.

The language of first-order logic with generalized quantifiers is obtained by adding two-place quantifier symbols to the language of first-order logic. If Q is a generalized quantifier symbol and φ and ψ are formulas, $Qx(\varphi,\psi)$ is a formula. (We shall be mainly interested in the case where φ and ψ are formulas of first-order logic.) The new clause in the Tarski style truth definition looks like the following. Let $\mathbf{M}$ be a model with domain M. For any quantifier symbol Q,

$$\begin{aligned}&\mathbf{M} \models Qx(\varphi,\psi)[s] \text{ iff}\\ &\qquad \langle \{\, a \in M \mid \mathbf{M} \models \varphi[s(a/x)]\,\}, \{\, a \in M \mid \mathbf{M} \models \psi[s(a/x)]\,\}\rangle \in Q^{\mathbf{M}}.\end{aligned}$$

$Q^{\mathbf{M}} \subseteq \mathrm{pow}(M) \times \mathrm{pow}(M)$ is the interpretation of Q in $\mathbf{M}$. $s(a/x)$ denotes the assignment s' such that $s'(x) = a$ and $s'(y) = s(y)$ for all $y \neq x$. From this definition, it follows that equivalent formulas are always intersubstitutable.

Equivalence (EQUI).

$$\forall x(\varphi \leftrightarrow \varphi') \wedge \forall x(\psi \leftrightarrow \psi') \rightarrow (Qx(\varphi,\psi) \leftrightarrow Qx(\varphi',\psi'))$$

Also, the choice of bound variable is arbitrary (with the usual provisos).

Renaming. $\qquad Qx(\varphi(x),\psi(x)) \leftrightarrow Qy(\varphi(y),\psi(y))$

For any quantifier Q, the above two schemata are always valid in first-order logic with generalized quantifiers.

We intend each Q to represent a natural language determiner. Consequently, $Q^{\mathbf{M}}$ is not just an arbitrary subset of $\mathrm{pow}(M) \times \mathrm{pow}(M)$. Interpretations of quantifier symbols are constrained by conditions to be satisfied within and across models.

Firstly, $Q^{\mathbf{M}}$ should depend just on M, the universe of $\mathbf{M}$. Thus, each Q is associated with a functional assigning to each non-empty set U a subset Q_U of $\mathrm{pow}(U) \times \mathrm{pow}(U)$. $Q^{\mathbf{M}}$ is set to Q_M. Moreover, Q_M's must 'agree' with each other in the sense that

$$\text{for all } A, B \subseteq M, N,\ \ Q_M AB \ \text{ iff } \ Q_N AB.$$

This condition is called *Extension*.[5] As a model-theoretic property of formulas in our language, it is expressed as follows:

Extension (EXT). For any $\mathbf{M}$, $\mathbf{N}$ and $s\colon VAR \to M \cap N$, if

$$\{\, a \in M \mid \mathbf{M} \models \varphi[s(a/x)] \,\} = \{\, a \in N \mid \mathbf{N} \models \varphi[s(a/x)] \,\}$$

and

$$\{\, a \in M \mid \mathbf{M} \models \psi[s(a/x)] \,\} = \{\, a \in N \mid \mathbf{N} \models \psi[s(a/x)] \,\},$$

then

$$\mathbf{M} \models Qx(\varphi, \psi)[s] \quad \text{iff} \quad \mathbf{N} \models Qx(\varphi, \psi)[s].$$

Conservativity is one of the most important properties of quantifiers and it has been claimed to hold universally of all natural language determiners (Barwise and Cooper 1981, Keenan and Stavi 1986). At the level of denotation, it says

$$\text{For all } A, B \subseteq M,\ \ Q_M AB \ \text{ iff } \ Q_M A(A \cap B).$$

The corresponding formula is the following:

Conservativity (CONS).

$$Qx(\varphi, \psi) \leftrightarrow Qx(\varphi, \varphi \wedge \psi).$$

EXT and CONS are universal principles that are supposed to hold of all natural language quantifiers. For 'logical' quantifiers, another principle called Quantity (QUANT) is usually assumed, which says (under CONS and EXT) that $Q_M AB$ depends just on the size of $A - B$ and $A \cap B$. QUANT does not play any role in this work. Henceforth, EXT and CONS will always be assumed, if Q is supposed to represent a natural language determiner.

In addition to general properties like EXT and CONS, there are special properties of specific quantifiers that are of interest. *Monotonicity* properties are the focus of the present paper. In terms of denotations, they are:

[5] It is called Extension since it is equivalent to the condition restricted to the case $M \subseteq N$.

$\uparrow$MON	for all $A, A', B \subseteq M$, $Q_M AB$ and $A \subseteq A'$ imply $Q_M A'B$
$\downarrow$MON	for all $A, A', B \subseteq M$, $Q_M AB$ and $A' \subseteq A$ imply $Q_M A'B$
MON$\uparrow$	for all $A, B, B' \subseteq M$, $Q_M AB$ and $B \subseteq B'$ imply $Q_M AB'$
MON$\downarrow$	for all $A, B, B' \subseteq M$, $Q_M AB$ and $B' \subseteq B$ imply $Q_M AB'$

In the language of first-order logic with generalized quantifiers, they are expressed by the following formulas:

Monotonicity.

$\uparrow$MON	$\forall x(\varphi \to \varphi') \to (Qx(\varphi, \psi) \to Qx(\varphi', \psi))$
$\downarrow$MON	$\forall x(\varphi' \to \varphi) \to (Qx(\varphi, \psi) \to Qx(\varphi', \psi))$
MON$\uparrow$	$\forall x(\psi \to \psi') \to (Qx(\varphi, \psi) \to Qx(\varphi, \psi'))$
MON$\downarrow$	$\forall x(\psi' \to \psi) \to (Qx(\varphi, \psi) \to Qx(\varphi, \psi'))$

Since CONS gives the left argument of a quantifier a privileged role, left monotonicity ($\uparrow$MON, $\downarrow$MON) and right monotonicity (MON$\uparrow$, MON$\downarrow$) turn out to be very different properties. An illustration of the difference is given by the following model-theoretic characterization of left monotonicity, adapted from Westerståhl 1989 (p. 79).

PROPOSITION 1. Assume that Q obeys EXT and CONS. Then Q is $\uparrow$MON ($\downarrow$MON) if and only if $Qx(P(x), R(x))$ is preserved under extensions (submodels).[6]

The generalization of Proposition 1 will be of our interest (Section 4.5).

3 Dynamic Predicate Logic

Dynamic predicate logic of Groenendijk and Stokhof (1991) is presented as an alternative interpretation of the language of first-order logic. It is more convenient for our purposes to present DPL as an *extension* of first-order logic. We provide all necessary definitions, but cannot fully convey the intuitions behind the system. For that, the reader is referred to Groenendijk and Stokhof 1991.

The language of DPL contains, in addition to equality, relation symbols, function symbols, constant symbols, variables, and *static connectives*

$$\neg,\ \wedge,\ \vee,\ \to,\ \leftrightarrow,\ \forall,\ \exists$$

[6] $\mathbf{M}$ is called a submodel of $\mathbf{N}$ ($\mathbf{M} \subseteq \mathbf{N}$) if $M \subseteq N$ and $P^{\mathbf{M}} = P^{\mathbf{N}} \upharpoonright M^n$ (restriction of $P^{\mathbf{N}}$ to M^n) for all n-ary relation symbols P, $F^{\mathbf{M}} = F^{\mathbf{N}} \upharpoonright M^n$ for all n-ary function symbols F, and $c^{\mathbf{M}} = c^{\mathbf{N}} \in M$ for all constant symbols c. (In Proposition 1, all that matters is the interpretation of P and R.) If $\mathbf{M} \subseteq \mathbf{N}$, $\mathbf{N}$ is called an extension of $\mathbf{M}$. A sentence φ is said to be preserved under extensions (submodels) if $\mathbf{M} \models \varphi$ implies $\mathbf{N} \models \varphi$ whenever $\mathbf{M} \subseteq \mathbf{N}$ ($\mathbf{N} \subseteq \mathbf{M}$).

from first-order logic, *dynamic connectives*

$$;,\ \Rightarrow,\ \mathscr{E}$$

(dynamic conjunction, dynamic implication, and dynamic existential quantifier, respectively). Moreover, I include *'meta' connectives*

$$\simeq,\ \preceq,$$

which are interpreted like Groenendijk and Stokhof's equivalence and meaning inclusion relativized to models and assignments.

The semantics of static connectives can be completely explained in terms of the usual satisfaction conditions:

$$\mathbf{M} \models \varphi[s]$$

(s satisfies φ in $\mathbf{M}$) just as in first-order logic. In contrast, the semantics of dynamic and 'meta' connectives must essentially rely on more 'dynamic' *transition conditions*:

$$s\,[\![\varphi]\!]_{\mathbf{M}}\,s'.$$

In Table 2, we give the semantics of DPL as a simultaneous recursive definition of $\mathbf{M} \models \varphi[s]$ and $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$. The notions of models and assignments, as well as the interpretation $t^{\mathbf{M},s}$ of a term t with respect to a model $\mathbf{M}$ and an assignment s are the familiar ones from first-order logic.

The notions of truth and validity are defined in the usual way in terms of satisfaction: $\mathbf{M} \models \varphi$ (φ is true in $\mathbf{M}$) if and only if for every $s\colon VAR \to M$, $\mathbf{M} \models \varphi[s]$; and $\models \varphi$ (φ is valid) if and only if for all $\mathbf{M}$, $\mathbf{M} \models \varphi$.

Note that for 1–9, the definition of $\mathbf{M} \models \varphi[s]$ is identical to the usual one in first-order logic. Consequently, if φ is a first-order formula, $\mathbf{M} \models \varphi[s]$ in DPL if and only if $\mathbf{M} \models \varphi[s]$ in first-order logic. In this sense, we can say our version of DPL is an extension of first-order logic.

Let us call $[\varphi]_{\mathbf{M}} = \{\, s \mid \mathbf{M} \models \varphi[s] \,\}$ the *static denotation* of φ (in $\mathbf{M}$), and $[\![\varphi]\!]_{\mathbf{M}} = \{\, \langle s, s' \rangle \mid s\,[\![\varphi]\!]_{\mathbf{M}}\,s' \,\}$ the *dynamic denotation* of φ (in $\mathbf{M}$). It is easy to see $[\varphi]_{\mathbf{M}} = \mathrm{dom}([\![\varphi]\!]_{\mathbf{M}})$ (the domain of $[\![\varphi]\!]_{\mathbf{M}}$) for every formula φ; i.e., the static denotation is always recoverable from the dynamic denotation. For some formulas φ, their dynamic denotation can be extracted from their static denotation—$[\![\varphi]\!]_{\mathbf{M}} = \mathrm{id} \restriction [\varphi]_{\mathbf{M}}$ (the identity relation restricted to $[\varphi]_{\mathbf{M}}$) holds. Such formulas are called *tests*.

For 1–9, 11, 13, and 14, the definition of $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$ is the same—$[\![\varphi]\!]_{\mathbf{M}} = \mathrm{id} \restriction [\varphi]_{\mathbf{M}}$. Formulas of these forms are always tests and have no external dynamic effect. For this reason, all connectives except ; and $\mathscr{E}$ are called *externally static*. The static connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$) from fisrt-order logic are also *internally static*. If the main connective of φ is one of these, the static, and hence dynamic, denotation of φ can be calculated from the static denotation of its immediate subformula(s). In contrast, the dynamic

	φ	$\mathbf{M} \models \varphi[s]$ iff ...	$s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$ iff ...
1.	$t_1 = t_2$	$t_1^{\mathbf{M},s} = t_2^{\mathbf{M},s}$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
2.	$R(t_1,\dots,t_n)$	$\langle t_1^{\mathbf{M},s},\dots,t_n^{\mathbf{M},s}\rangle \in R^{\mathbf{M}}$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
3.	$\neg\psi$	$\mathbf{M} \not\models \psi[s]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
4.	$\psi \wedge \chi$	$\mathbf{M} \models \psi[s]$ and $\mathbf{M} \models \chi[s]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
5.	$\psi \vee \chi$	$\mathbf{M} \models \psi[s]$ or $\mathbf{M} \models \chi[s]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
6.	$\psi \rightarrow \chi$	$\mathbf{M} \models \psi[s]$ implies $\mathbf{M} \models \chi[s]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
7.	$\psi \leftrightarrow \chi$	$\mathbf{M} \models \psi[s]$ iff $\mathbf{M} \models \chi[s]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
8.	$\forall x\psi$	for all $a \in M$, $\mathbf{M} \models \psi[s(a/x)]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
9.	$\exists x\psi$	for some $a \in M$, $\mathbf{M} \models \psi[s(a/x)]$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
10.	$\psi\,;\chi$	for some s', $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$	for some s'', $s\,[\![\psi]\!]_{\mathbf{M}}\,s''$ and $s''\,[\![\chi]\!]_{\mathbf{M}}\,s'$
11.	$\psi \Rightarrow \chi$	for all s', $s\,[\![\psi]\!]_{\mathbf{M}}\,s'$ implies $\mathbf{M} \models \chi[s']$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
12.	$\mathscr{E}x\psi$	for some s', $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$	for some $a \in M$, $s(a/x)\,[\![\psi]\!]_{\mathbf{M}}\,s'$
13.	$\psi \simeq \chi$	for all s', $s\,[\![\psi]\!]_{\mathbf{M}}\,s'$ iff $s\,[\![\chi]\!]_{\mathbf{M}}\,s'$	$s = s'$ and $\mathbf{M} \models \varphi[s]$
14.	$\psi \preceq \chi$	for all s', $s\,[\![\psi]\!]_{\mathbf{M}}\,s'$ implies $s\,[\![\chi]\!]_{\mathbf{M}}\,s'$	$s = s'$ and $\mathbf{M} \models \varphi[s]$

Table 2: Semantics of DPL.

connectives (;, $\Rightarrow$, $\mathscr{E}$) and our 'meta' connectives ($\simeq$, $\preceq$) are *internally dynamic*. To calculate the static and dynamic denotations of a formula whose main connective is ;, $\Rightarrow$, $\simeq$, or $\preceq$, the dynamic denotation of its immediate subformulas must be consulted. In the case of $\mathscr{E}$, $[\mathscr{E}x\psi]_{\mathbf{M}}$ is determined by $[\psi]_{\mathbf{M}}$ ($[\mathscr{E}x\psi]_{\mathbf{M}} = [\exists x\psi]_{\mathbf{M}}$), but to calculate $[\![\mathscr{E}x\psi]\!]_{\mathbf{M}}$ one must look at $[\![\psi]\!]_{\mathbf{M}}$. The connectives ; and $\mathscr{E}$ are even *externally dynamic*. If the main connective of φ is ; or $\mathscr{E}$, $[\![\varphi]\!]_{\mathbf{M}}$ is not determined by $[\varphi]_{\mathbf{M}}$ in general, and s and s' such that $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$ can be different.[7]

[7] We note that our use of the term 'internally dynamic' is slightly different from Groenendijk and Stokhof's. Since $\simeq$ and $\preceq$ create no new variable binding, they would not be internally dynamic connectives in their sense. On the other hand, their usage would make our $\exists$ internally dynamic. (They do not have a precise definition,

A remark on our 'meta' connectives $\simeq$ and $\preceq$ is in order. Groenendijk and Stokhof (1991) use $\simeq$ and $\preceq$ as symbols in the metalanguage; in their paper, '$\varphi \simeq \psi$' and '$\varphi \preceq \psi$' mean 'for all $\mathbf{M}$, $[\![\varphi]\!]_{\mathbf{M}} = [\![\psi]\!]_{\mathbf{M}}$' and 'for all $\mathbf{M}$, $[\![\varphi]\!]_{\mathbf{M}} \subseteq [\![\psi]\!]_{\mathbf{M}}$', respectively. If we write $s\,[\![\varphi]\!]_{\mathbf{M}}$ for $\{\, s' \mid s\,[\![\varphi]\!]_{\mathbf{M}}\, s' \,\}$,

$$\begin{array}{rcl} \mathbf{M} \models \varphi \simeq \psi[s] & \text{iff} & s\,[\![\varphi]\!]_{\mathbf{M}} = s\,[\![\varphi]\!]_{\mathbf{M}}, \\ \mathbf{M} \models \varphi \preceq \psi[s] & \text{iff} & s\,[\![\varphi]\!]_{\mathbf{M}} \subseteq s\,[\![\varphi]\!]_{\mathbf{M}}, \end{array}$$

$$\begin{array}{rcl} \mathbf{M} \models \varphi \simeq \psi & \text{iff} & [\![\varphi]\!]_{\mathbf{M}} = [\![\varphi]\!]_{\mathbf{M}}, \\ \mathbf{M} \models \varphi \preceq \psi & \text{iff} & [\![\varphi]\!]_{\mathbf{M}} \subseteq [\![\varphi]\!]_{\mathbf{M}}, \end{array}$$

and

$$\begin{array}{rcl} \models \varphi \simeq \psi & \text{iff} & \text{for all } \mathbf{M},\ [\![\varphi]\!]_{\mathbf{M}} = [\![\varphi]\!]_{\mathbf{M}}, \\ \models \varphi \preceq \psi & \text{iff} & \text{for all } \mathbf{M},\ [\![\varphi]\!]_{\mathbf{M}} \subseteq [\![\varphi]\!]_{\mathbf{M}}. \end{array}$$

Thus, Groenendijk and Stokhof's '$\varphi \simeq \psi$' and '$\varphi \preceq \psi$' are our '$\models \varphi \simeq \psi$' and '$\models \varphi \preceq \psi$'. This justifies our own use of their symbols. $\simeq$ and $\preceq$ will be useful in expressing certain principles, and are not intended for use in representing natural language sentences. In what follows, metavariables like φ and ψ will always range over formulas without $\simeq$ or $\preceq$. Note that $\varphi \simeq \psi$ is equivalent to $(\varphi \preceq \psi) \wedge (\psi \preceq \varphi)$.

There is an obvious correspondence between dynamic connectives and their static counterparts. If φ and ψ are tests, the following equivalences are valid:

(9) $$\varphi\,;\psi \quad \simeq \quad \varphi \wedge \psi$$

(10) $$\varphi \Rightarrow \chi \quad \simeq \quad \varphi \rightarrow \chi$$

(In (10), χ does not have to be a test.) Also, for any φ,[8]

(11) $$\mathscr{E}x\varphi \quad \leftrightarrow \quad \exists x\varphi$$

However, even if φ is a test,

$$\mathscr{E}x\varphi \quad \simeq \quad \exists x\varphi$$

usually does not hold. Although ; just passes on, so to speak, the external dynamic force of its conjuncts, part of the external dynamic force of $\mathscr{E}x\varphi$ is created by $\mathscr{E}x$.

however.) We can state our definition in the following way. A connective C is externally static if a formula with C as its main connective is always a test. Otherwise, C is externally dynamic. C is internally dynamic if the dynamic denotation of a formula with C as its main connective cannot in general be determined by the static denotation of its immediate subformula(s). Otherwise, C is internally static.

[8] Henceforth, we may just assert a formula to mean it is valid.

The 'meta' connectives $\simeq$ and $\preceq$ correspond to $\leftrightarrow$ and $\rightarrow$. If φ and ψ are tests,

$$\begin{array}{rcl}(\varphi \simeq \psi) & \simeq & (\varphi \leftrightarrow \psi) \\ (\varphi \preceq \psi) & \simeq & (\varphi \rightarrow \psi)\end{array}$$

Note also that for any φ and ψ,

$$\begin{array}{rcl}(\varphi \simeq \psi) & \rightarrow & (\varphi \leftrightarrow \psi) \\ (\varphi \preceq \psi) & \rightarrow & (\varphi \rightarrow \psi)\end{array}$$

Both $\Rightarrow$ and $\preceq$ correspond to $\rightarrow$. If φ and ψ are tests, $\varphi \Rightarrow \psi$ and $\varphi \preceq \psi$ are equivalent. Intuitively, the difference between $\Rightarrow$ and $\preceq$ is that the semantics of the former is 'sequential' while that of the latter is 'parallel'.

Note that $\Rightarrow$ and the static connectives except $\neg$ are definable in terms of $\neg$, ;, and $\mathscr{E}$, using the equivalences:

$$\begin{array}{rcl}\varphi \Rightarrow \psi & \simeq & \neg(\varphi\,;\neg\psi) \\ \varphi \rightarrow \psi & \simeq & \neg\psi \Rightarrow \neg\varphi \\ \forall x\varphi & \simeq & \neg\mathscr{E}x\neg\varphi\end{array}$$

and the standard equivalences in first-order logic. See Groenendijk and Stokhof 1991 for details.

The most important properties of dynamic connectives are expressed in the following equivalences:

(12) $$(\varphi\,;\psi)\,;\chi \simeq \varphi\,;(\psi\,;\chi)$$

(13) $$\mathscr{E}x\varphi\,;\psi \simeq \mathscr{E}x(\varphi\,;\psi)$$

(14) $$\mathscr{E}x\varphi \Rightarrow \psi \simeq \forall x(\varphi \Rightarrow \psi)$$

In the last two schemata, (13) and (14), there is no restriction on ψ; unlike the corresponding equivalences in first-order logic, x can occur free in ψ.[9] These properties of dynamic connectives are used to represent intersentential and donkey anaphora in natural language. For example, one may translate

If Pedro owns a donkey, it is kept in the barn

in DPL as

$$\mathscr{E}x(\text{donkey}(x)\,;\text{own}(\text{Pedro},x)) \Rightarrow \text{kept-in-the-barn}(x).$$

[9] In first-order logic, if x does not occur free in ψ, $\exists x\varphi \wedge \psi$ and $\exists x\varphi \rightarrow \psi$ are equivalent to $\exists x(\varphi \wedge \psi)$ and $\forall x(\varphi \rightarrow \psi)$, respectively. If x is free in ψ, the equivalences do not hold in general.

The latter can be seen to be equivalent to

$$\forall x(\text{donkey}(x) \wedge \text{own}(\text{Pedro}, x) \rightarrow \text{kept-in-the-barn}(x)).$$

For more examples and discussion, see Groenendijk and Stokhof 1990, 1991.[10]

Groenendijk and Stokhof (1991) define $\text{AQV}(\varphi)$, the set of *active quantifier variables* of φ, and $\text{FV}(\varphi)$, the set of free variables in φ. $\text{AQV}(\varphi)$ will be very important in what follows. We define $\text{AQV}(\varphi)$ as follows. An occurrence of a dynamic existential quantifier in φ is called *potentially active* if it does not lie within the scope of any externally static connective in φ. For any variable x, the rightmost potentially active occurrence of $\mathscr{E}x$ in φ is called an *active* occurrence of $\mathscr{E}x$. $\text{AQV}(\varphi)$, then, is the set of variables x such that there is an active (or, equivalently, potentially active) occurrence of $\mathscr{E}x$ in φ. $\text{FV}(\varphi)$ is defined in terms of another useful notion, $\text{fv}(\varphi)$, the set of free occurrences of variables in φ. Table 3 gives the definition of $\text{fv}(\varphi)$.[11] Like Groenendijk and Stokhof, we allow ourselves to be sloppy by not having an explicit way of referring to occurrences. $\text{FV}(\varphi)$ is then defined to be the set of variables x such that there is an occurrence of x in $\text{fv}(\varphi)$.

If φ is a first-order formula, $\text{AQV}(\varphi) = \emptyset$ and $\text{FV}(\varphi)$ is the set of free variables in φ in the usual sense.

The following facts point to the 'meaning' of $\text{AQV}(\varphi)$ and $\text{FV}(\varphi)$:

- If for some $\mathbf{M}$, there are s and s' such that $s\,[\![\varphi]\!]_{\mathbf{M}}\,s'$ and $s(x) \neq s'(x)$, then $x \in \text{AQV}(\varphi)$.

- If $x \notin \text{FV}(\varphi)$, then for all $\mathbf{M}$, for all $s\colon \mathit{VAR} \rightarrow M$ and $a \in M$, $\mathbf{M} \models \varphi[s]$ iff $\mathbf{M} \models \varphi[s(a/x)]$.

[10] Groenendijk and Stokhof express the fact that (13) and (14) are valid by saying that the dynamic existential quantifier can bind variables outside its syntactic scope. For Groenendijk and Stokhof, free occurrences of x in the second conjunct of $\mathscr{E}x\varphi\,;\psi$ are bound by the dynamic existential quantifier in the first conjunct, and free occurrences of x in the consequent of $\mathscr{E}x\varphi \Rightarrow \psi$ are bound by the dynamic existential quantifier in the antecedent. This terminology might be misleading. In these formulas, $\mathscr{E}x$ does not bind the x in φ and the x in ψ in the same way. If anything, it is the combination of the dynamic existential quantifier and the dynamic main connective (; or $\Rightarrow$) that binds the x in ψ. (This will be more apparent when the main connective is a dynamic generalized quantifier.) It is clear that the occurrences of x in these formulas are not like free variables in first-order logic, and so I will follow Groenendijk and Stokhof in saying that the free occurrences of x in ψ are 'bound' in $\mathscr{E}x\varphi\,;\psi$ or $\mathscr{E}x\varphi \Rightarrow \psi$. I will avoid, however, the terminology 'bound by $\mathscr{E}x$' in such cases; the 'binding' relation between x and $\mathscr{E}x$ is an indirect one mediated by ; or $\Rightarrow$. Cf. Barwise's (1987) three-way distinction between *captured*, *restrained*, and *free*.

[11] We do not include clauses for 'meta' connectives, as the notion of free variables sometimes does not quite behave as expected in the presence of $\simeq$ or $\preceq$.

	φ	$\mathrm{fv}(\varphi)$
1.	$t_1 = t_2$	all occurrences of variables in φ
2.	$R(t_1, \ldots, t_n)$	all occurrences of variables in φ
3.	$\neg\psi$	$\mathrm{fv}(\psi)$
4.	$\psi \wedge \chi$	$\mathrm{fv}(\psi) \cup \mathrm{fv}(\chi)$
5.	$\psi \vee \chi$	$\mathrm{fv}(\psi) \cup \mathrm{fv}(\chi)$
6.	$\psi \to \chi$	$\mathrm{fv}(\psi) \cup \mathrm{fv}(\chi)$
7.	$\psi \leftrightarrow \chi$	$\mathrm{fv}(\psi) \cup \mathrm{fv}(\chi)$
8.	$\forall x\psi$	$\mathrm{fv}(\psi)$ minus all occurrences of x
9.	$\exists x\psi$	$\mathrm{fv}(\psi)$ minus all occurrences of x
10.	$\psi ; \chi$	$\mathrm{fv}(\psi) \cup \{ x \in \mathrm{fv}(\chi) \mid x \notin \mathrm{AQV}(\psi) \}$
11.	$\psi \Rightarrow \chi$	$\mathrm{fv}(\psi) \cup \{ x \in \mathrm{fv}(\chi) \mid x \notin \mathrm{AQV}(\psi) \}$
12.	$\mathscr{E} x\psi$	$\mathrm{fv}(\psi)$ minus all occurrences of x

Table 3: Definition of $\mathrm{fv}(\varphi)$.

As is already clear by now, the dynamic existential quantifier $\mathscr{E}$ plays a pivotal role in DPL. All 'dynamics' of DPL formulas originate in active occurrences of $\mathscr{E}$. If $\mathrm{AQV}(\varphi) = \emptyset$, then φ is a test (but not necessarily conversely).[12]

In a certain sense, the semantics of $\mathscr{E}$ can be regarded as a reconciliation of two competing views on indefinite noun phrases in natural language: the traditional idea of 'indefinites as existential quantifiers' and the idea of 'indefinites as variables' made popular by discourse representation theory. For example, consider the formula $\mathscr{E} xP(x)$. One can easily see

$$\begin{aligned} \mathrm{dom}([\![\mathscr{E} xP(x)]\!]_{\mathbf{M}}) &= [\exists xP(x)]_{\mathbf{M}}, \\ \mathrm{ran}([\![\mathscr{E} xP(x)]\!]_{\mathbf{M}}) &= [P(x)]_{\mathbf{M}}. \end{aligned}$$

Let us make this observation more general. To make matters simple, we confine ourselves to formulas with certain desirable syntactic properties. In what follows, let φ range over formulas which satisfy the following two conditions:

(15) $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$.

(16) For any variable x, there is at most one potentially active occurrence of $\mathscr{E} x$ in φ.

[12]Formulas φ with $\mathrm{AQV}(\varphi) = \emptyset$ correspond to what Groenendijk and Stokhof (1991) call *conditions*. They make a false statement that φ is a test iff φ is a condition or a contradiction (FACT 6, p. 58). A counterexample is $x = a ; \mathscr{E} x(x = a)$, which is a test, but not a condition or a contradiction. $[\![x = a ; \mathscr{E} x(x = a)]\!]_{\mathbf{M}} = [\![x = a]\!]_{\mathbf{M}}$. Their claim holds when confined to the case $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$.

(15) excludes formulas like $P(x) ; \mathscr{E}xR(x)$, and (16) excludes formulas like $\mathscr{E}x(P(x) ; \mathscr{E}xR(x))$. Neither restriction, however, essentially reduces the expressive power of the language.[13] Now define φ^* to be the result of erasing all (potentially) active occurrences of dynamic existential quantifiers in φ. For example,

$$\begin{aligned}&(\text{farmer}(x) ; \mathscr{E}y(\text{donkey}(y) ; \text{own}(x,y)))^* \\ &\quad = \text{farmer}(x) ; (\text{donkey}(y) ; \text{own}(x,y)).\end{aligned}$$

Then

$$\begin{aligned}\text{dom}([\![\varphi]\!]_{\mathbf{M}}) &= [\exists x_1 \ldots \exists x_n \varphi^*]_{\mathbf{M}}, \\ \text{ran}([\![\varphi]\!]_{\mathbf{M}}) &= [\varphi^*]_{\mathbf{M}},\end{aligned}$$

where $\{x_1, \ldots, x_n\} = \text{AQV}(\varphi)$.[14]

Here are some more important equivalences.

(17) $$\varphi ; \psi \Rightarrow \chi \quad \simeq \quad \varphi \Rightarrow (\psi \Rightarrow \chi)$$

If $x \notin \text{FV}(\varphi) \cup \text{AQV}(\varphi)$,

(18) $$\varphi ; \mathscr{E}x\psi \quad \simeq \quad \mathscr{E}x(\varphi ; \psi)$$

If $\text{AQV}(\varphi) \cap \text{FV}(\psi) = \emptyset$,

(19) $$\varphi ; \psi \quad \leftrightarrow \quad \varphi \wedge \psi$$

(20) $$\varphi \Rightarrow \psi \quad \simeq \quad \varphi \rightarrow \psi$$

(19) and (20) generalize (9) and (10). The 'internal dynamics' of ; and $\Rightarrow$ is exhausted by their capacity to mediate indirect binding discussed above; if $\text{AQV}(\varphi) \cap \text{FV}(\varphi) = \emptyset$, this does not happen, and ; and $\Rightarrow$ collapse to $\wedge$ and $\rightarrow$ (as far as the satisfaction conditions are concerned, in the case of ;).

Replacement of equivalents in DPL. In first-order logic, equivalent formulas are intersubstitutable *salva veritate*. Formally, if $\varphi(\psi)$ is a first-order

[13] For any formula ψ, a formula φ which meets (15) and (16) such that $\varphi \leftrightarrow \psi$ can be obtained by mere renaming of variables. (15) is discussed by Groenendijk and Stokhof (1991). Formulas which violate it are arguably never necessary to represent natural language sentences and discourses. As for (16), note that, for any formula ψ satisfying (15), there is a φ satisfying both (15) and (16) such that $\varphi \simeq \psi$. For example,

$$\mathscr{E}x(P(x) ; \mathscr{E}xR(x)) \quad \simeq \quad \exists xP(x) ; \mathscr{E}xR(x)$$

[14] By convention, the notation φ^* will always presuppose that φ satisfies (15) and (16) in the remainder of the paper.

formula with subformula ψ, and $\{x_1,\ldots,x_n\}$ are the variables which have occurrences in $(\mathrm{fv}(\psi)-\mathrm{fv}(\varphi(\psi)))\cup(\mathrm{fv}(\psi')-\mathrm{fv}(\varphi(\psi')))$,

(21) $$\forall x_1\ldots x_n(\psi\leftrightarrow\psi')\rightarrow(\varphi(\psi)\leftrightarrow\varphi(\psi'))$$

An analogue of (21) in DPL is the following: if $\varphi(\psi)$ and ψ' are DPL formulas (without 'meta' connectives) and $\{x_1,\ldots,x_n\}$ are the variables which have occurrences in $(\mathrm{fv}(\psi)-\mathrm{fv}(\varphi(\psi)))\cup(\mathrm{fv}(\psi')-\mathrm{fv}(\varphi(\psi')))$,

(22) $$\forall x_1\ldots x_n(\psi\simeq\psi')\rightarrow(\varphi(\psi)\simeq\varphi(\psi'))$$

(In case $\varphi(\psi)$ and $\varphi(\psi')$ contain no free variables, this amounts to

$$\forall(\psi\simeq\psi')\rightarrow(\varphi(\psi)\simeq\varphi(\psi')),$$

where $\forall$ indicates universal closure.) In particular, we note the validity of the following. Let $\{x_1,\ldots,x_n\}=\mathrm{AQV}(\psi)$.

(23) $$\psi\simeq\psi'\wedge\forall x_1\ldots x_n(\chi\simeq\chi')\quad\rightarrow\quad(\psi;\chi\;\simeq\;\psi';\chi')$$

(24) $$\psi\simeq\psi'\wedge\forall x_1\ldots x_n(\chi\leftrightarrow\chi')\quad\rightarrow\quad(\psi\Rightarrow\chi\;\simeq\;\psi'\Rightarrow\chi')$$

(25) $$\forall x(\psi\simeq\psi')\quad\rightarrow\quad(\mathscr{E}x\psi\simeq\mathscr{E}x\psi')$$

With this much background, we are now ready to introduce generalized quantifiers into dynamic predicate logic.

4 Generalized Quantifiers in Dynamic Predicate Logic

We can introduce static generalized quantifiers into dynamic predicate logic in the obvious way. If $\varphi=Qx(\psi,\chi)$,

$$\begin{aligned}&\mathbf{M}\models\varphi[s]\ \text{ iff}\\ &\qquad\langle\{\,a\in M\mid\mathbf{M}\models\psi[s(a/x)]\,\},\{\,a\in M\mid\mathbf{M}\models\chi[s(a/x)]\,\}\rangle\in Q_M,\\ &s\,[\![\varphi]\!]_{\mathbf{M}}\,s'\ \text{ iff }\ s=s'\text{ and }\mathbf{M}\models\varphi[s].\end{aligned}$$

This was the format for all static connectives in DPL.[15]

[15] The new clause for $\mathrm{fv}(\varphi)$ is:

	φ	$\mathrm{fv}(\varphi)$
13.	$Qx(\psi,\chi)$	$\mathrm{fv}(\psi)\cup\mathrm{fv}(\chi)$ minus all occurrences of x

Q is externally static, so if $\varphi=Qx(\psi,\chi)$, no occurrence of dynamic existential quantifier in φ is potentially active, and $\mathrm{AQV}(\varphi)=\emptyset$.

Since we built our version of DPL as an extension of first-order logic, DPL with static generalized quantifiers (DPL(Q)) in turn becomes an extension of first-order logic with generalized quantifiers. For any static generalized quantifier Q, Q behaves in DPL(Q) just like it does in first-order logic with generalized quantifiers. Basic principles like EQUI and Renaming remain valid in DPL(Q), metavariables like φ, ψ, etc., now ranging over formulas of DPL(Q). For Q representing natural language determiners, EXT and CONS continue to hold in DPL(Q), and depending on the kind of quantifier it is, one or more of the MON formulas remain valid.

To model donkey sentences with DPL, addition of static generalized quantifiers will not be enough; we need to introduce dynamic generalized quantifiers into DPL (DPL($Q, \mathscr{Q}$)). Such quantifiers must allow active dynamic existential quantifiers in their first argument to provide values for free variables in their second argument; i.e., they must be internally dynamic in the way ; and $\Rightarrow$ are. We may want some dynamic generalized quantifiers to be externally dynamic as well, to model a few natural language determiners that show an external dynamic effect like *some* and *a*. Nevertheless, since we are not concerned with the external dynamic effect of donkey sentences in this paper, we will ignore such possibilities, and work with internally dynamic but externally static generalized quantifiers. We use script $\mathscr{Q}$ as a symbol for such dynamic generalized quantifiers.

The definition of $\mathrm{fv}(\varphi)$ is naturally extended as follows:

	φ	$\mathrm{fv}(\varphi)$
14.	$\mathscr{Q}x(\psi,\chi)$	$\mathrm{fv}(\psi) \cup \{\, y \in \mathrm{fv}(\chi) \mid y \notin \mathrm{AQV}(\psi)\,\}$ minus all occurrences of x

Since $\mathscr{Q}$ is externally static, if $\varphi = \mathscr{Q}x(\psi,\chi)$, no occurrence of dynamic existential quantifier in φ is potentially active, and $\mathrm{AQV}(\varphi) = \emptyset$.

It is not a straightforward matter to give a general format for the semantics for dynamic generalized quantifiers as we did with static ones. It seems to me that there is some arbitrary choice to be made in assigning denotations to dynamic generalized quantifiers. For Chierchia (1990, 1992), dynamic generalized quantifiers are relations between dynamic properties, and the latter are functions from individuals to relations between assignments. Rooth (1987) takes relations between parametric sets, the latter being sets of individual-assignment pairs. Such semantic notions of dynamic generalized quantifiers may not be fully satisfactory, since possible arguments of such relations must be restricted (only values for finitely many variables should matter). It is like taking relations between sets of assignments as denotations of static generalized quantifiers. What we want is an analogue of relations between sets of individuals. (Use of partial assignments would improve the matter, however.)

It is possible to think of dynamic generalized quantifiers as a restricted class of polyadic quantifiers (of variable adicity). So a dynamic generalized quantifier $\mathscr{Q}$ can be associated with a family of two-place polyadic quantifiers $\mathscr{Q}^1_M$, $\mathscr{Q}^2_M$, ..., where $\mathscr{Q}^n_M$ is a binary relation between n-ary relations on M. $\mathscr{Q}^n_M$'s must 'agree' with each other in some sense. Then we may define the satisfaction conditions for $\varphi = \mathscr{Q}x(\psi,\chi)$ thus:

$$\begin{aligned}&\mathbf{M} \models \varphi[s] \text{ iff}\\ &\langle \{ \langle a, b_1, \dots, b_n \rangle \mid s(a/x) [\![\psi]\!]_{\mathbf{M}} s(a/x, b_1/y_1, \dots, b_n/y_n) \},\\ &\qquad \{ \langle a, b_1, \dots, b_n \rangle \mid \mathbf{M} \models \chi[s(a/x, b_1/y_1, \dots, b_n/y_n)] \} \rangle \in \mathscr{Q}^{n+1}_M,\end{aligned}$$

where $\{y_1, \dots, y_n\} = \mathrm{AQV}(\psi)$.[16] The idea is to interpret $\mathscr{Q}x(\psi,\chi)$ like $Q^{n+1}xy_1 \dots y_n(\psi^*, \chi)$, where Q^{n+1} is a two-place $n+1$-adic generalized quantifier. (The latter formula is reminiscent of discourse representation theory.) However, this format is still too general, and reference to $\mathrm{AQV}(\psi)$ may be vexing.

Fortunately, we can develop a theory of dynamic generalized quantifiers largely without taking a stance on what the right notion of denotation is for dynamic generalized quantifiers. This is because properties of a dynamic generalized quantifier $\mathscr{Q}$ can be stated as properties of formulas $\mathscr{Q}x(\varphi,\psi)$, as we did with static quantifiers in Section 2. Moreover, particular dynamic generalized quantifiers we will consider are in fact the ones given by simple definitions in terms of static quantifiers and dynamic connectives of DPL.

Here, we note some obvious principles. As with static generalized quantifiers, dynamic generalized quantifiers allow renaming of the bound variable (with a fresh one):[17]

Renaming. $\quad \mathscr{Q}x(\varphi(x), \psi(x)) \leftrightarrow \mathscr{Q}y(\varphi(y), \psi(y))$

Since our dynamic generalized quantifiers are externally static, the above $\leftrightarrow$ can be replaced by $\simeq$.

The same should hold for variables that get bound indirectly:

Subordinate Renaming.

$$\mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}x(\varphi', \psi[z/y])$$

where $y \in \mathrm{AQV}(\varphi)$, z is a fresh variable, φ' is the result of replacing occurrences of y not free in φ by z, and $\psi[z/y]$ is the result of replacing free occurrences of y in ψ by z.

[16] $x \notin \mathrm{AQV}(\psi)$ is assumed. An alternative would be:

$$\begin{aligned}&\mathbf{M} \models \varphi[s] \text{ iff}\\ &\langle \{ \langle a, b_1, \dots, b_n \rangle \mid \exists s' (s(a/x) [\![\psi]\!]_{\mathbf{M}} s', s'(y_i) = b_i \, (1 \le i \le n)) \},\\ &\qquad \{ \langle a, b_1, \dots, b_n \rangle \mid \mathbf{M} \models \chi[s(a/x, b_1/y_1, \dots, b_n/y_n)] \} \rangle \in \mathscr{Q}^{n+1}_M,\end{aligned}$$

where $\{y_1, \dots, y_n\} = \mathrm{AQV}(\psi) \cap \mathrm{FV}(\chi)$.

[17] Here, we must have $x \notin \mathrm{AQV}(\varphi(x))$. ($y$ is fresh, so we also have $y \notin \mathrm{AQV}(\varphi(y))$.)

The principle of Equivalence takes the following form. If $\{y_1,\ldots,y_n\} = \mathrm{AQV}(\varphi)$,

Dynamic Equivalence (DEQUI).

$$\forall x(\varphi \simeq \varphi') \wedge \forall x \forall y_1 \ldots \forall y_n(\psi \leftrightarrow \psi') \to (\mathcal{Q}x(\varphi,\psi) \leftrightarrow \mathcal{Q}x(\varphi',\psi'))$$

The dynamic effects of dynamic existential quantifiers in the first argument of $\mathcal{Q}$ reach into the second argument. This requires the extra quantifiers $\forall y_1 \ldots \forall y_n$ in the second conjunct of the antecedent. Just as with dynamic implication of DPL, internal dynamics of dynamic generalized quantifiers goes only one way; the dynamic effects of the second argument does not matter, whence only the static equivalence $\leftrightarrow$ is required with respect to the second argument. (Compare (24).)

The above three principles always hold for internally dynamic but externally static generalized quantifiers.

4.1 Chierchia's Two Definitions of Dynamic Generalized Quantifiers

Here we follow Chierchia (1990) and explore ways of *defining* dynamic generalized quantifiers within the resources of DPL with static generalized quantifiers. Chierchia (1990) has two schemata for defining a dynamic generalized quantifier in terms of a static counterpart. One schema is used to represent the weak reading of donkey sentences, and the other is used to represent the strong reading. In our present set-up of dynamic predicate logic with generalized quantifiers, they come out as follows. For any static generalized quantifier Q, define two dynamic generalized quantifiers $\mathcal{Q}_W$ and $\mathcal{Q}_S$ by:[18]

(26) $$\mathcal{Q}_W x(\varphi,\psi) \quad \leftrightarrow \quad Qx(\varphi, \varphi\,;\psi)$$

(27) $$\mathcal{Q}_S x(\varphi,\psi) \quad \leftrightarrow \quad Qx(\varphi, \varphi \Rightarrow \psi)$$

(26) is for the weak reading, and (27) is for the strong reading.[19]

Let us work out an example:

(28) Most farmers who own a donkey beat it.

[18] Chierchia's (1990) own schemata are cast in a higher-order framework. Also, for some mysterious reason, he does not use $\Rightarrow$ in (27), and instead uses an 'adverb of quantification' of universal force. It is clear that $\Rightarrow$ does the same job. Essentially the same definitions as (26) and (27) are also found in van Eijck and de Vries 1992.

[19] Note that if Q satisfies CONS, and if $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\psi) = \emptyset$,

$$\mathcal{Q}_W x(\varphi,\psi) \quad \leftrightarrow \quad \mathcal{Q}_S x(\varphi,\psi) \quad \leftrightarrow \quad Qx(\varphi,\psi).$$

The first argument of *most*, *farmers who own a donkey*, is translated into DPL as:

(29) $$\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)).$$

The second argument, *beat it*, is translated as:

(30) $$\text{beat}(x,y).$$

The two readings of (28) are represented by the following two formulas:

(31) $$\mathscr{MOST}_W x(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)),\,\text{beat}(x,y))$$

(32) $$\mathscr{MOST}_S x(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)),\,\text{beat}(x,y))$$

By definition, (31) and (32) are equivalent to (33) and (34), respectively.

(33) $$\begin{aligned}&MOST x(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)),\\ &\quad(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)))\,;\text{beat}(x,y))\end{aligned}$$

(34) $$\begin{aligned}&MOST x(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)),\\ &\quad(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)))\Rightarrow\text{beat}(x,y))\end{aligned}$$

By several validities in DPL mentioned earlier, we get

$$\begin{aligned}&\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y))\\ &\quad\leftrightarrow\quad\text{farmer}(x)\wedge\exists x(\text{donkey}(y)\wedge\text{own}(x,y)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)))\,;\text{beat}(x,y)\\ &\quad\leftrightarrow\quad\text{farmer}(x)\wedge\exists y(\text{donkey}(y)\wedge\text{own}(x,y)\wedge\text{beat}(x,y))\end{aligned}$$

and

$$\begin{aligned}&(\text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y)))\Rightarrow\text{beat}(x,y)\\ &\quad\leftrightarrow\quad\text{farmer}(x)\to\forall y(\text{donkey}(y)\wedge\text{own}(x,y)\to\text{beat}(x,y)).\end{aligned}$$

Using EQUI, then, we see that (33) and (34) are equivalent to (35) and (36), respectively:

(35) $$\begin{aligned}&MOST x(\text{farmer}(x)\wedge\exists y(\text{donkey}(y)\wedge\text{own}(x,y)),\\ &\quad\text{farmer}(x)\wedge\exists y(\text{donkey}(y)\wedge\text{own}(x,y)\wedge\text{beat}(x,y)))\end{aligned}$$

(36) $$\begin{aligned}&MOST x(\text{farmer}(x)\wedge\exists y(\text{donkey}(y)\wedge\text{own}(x,y)),\\ &\quad\text{farmer}(x)\to\forall y(\text{donkey}(y)\wedge\text{own}(x,y)\to\text{beat}(x,y)))\end{aligned}$$

By CONS, (35) and (36) in turn are seen to be equivalent to (37) and (38), respectively.

(37) $$MOST x(\text{farmer}(x) \wedge \exists y(\text{donkey}(y) \wedge \text{own}(x,y)), \exists y(\text{donkey}(y) \wedge \text{own}(x,y) \wedge \text{beat}(x,y)))$$

(38) $$MOST x(\text{farmer}(x) \wedge \exists y(\text{donkey}(y) \wedge \text{own}(x,y)), \forall y(\text{donkey}(y) \wedge \text{own}(x,y) \rightarrow \text{beat}(x,y)))$$

Note that (37) and (38) correspond to the two paraphrases *Most farmers who own a donkey beat a donkey they own* and *Most farmers who own a donkey beat every donkey they own* of (28), respectively.

The equivalences like these hold in general. If Q is conservative, $x, y \notin \text{AQV}(\chi(x,y))$, and $x \notin \text{AQV}(\varphi(x))$,

(39) $$\mathscr{Q}_W x(\varphi(x); \mathscr{E} y \chi(x,y),\ \psi(x,y)) \leftrightarrow Qx(\varphi(x) \wedge \exists y \chi(x,y), \exists y(\chi(x,y) \wedge \psi(x,y)))$$

(40) $$\mathscr{Q}_S x(\varphi(x); \mathscr{E} y \chi(x,y),\ \psi(x,y)) \leftrightarrow Qx(\varphi(x) \wedge \exists y \chi(x,y), \forall y(\chi(x,y) \rightarrow \psi(x,y)))$$

The formulas on the right-hand side of (39) and (40) are schematic representations of the weak and strong readings of donkey sentences of the form (1a) in first-order logic with generalized quantifiers.

Thus, the problem of choosing between the weak and strong readings of donkey sentences becomes the problem of choosing between $\mathscr{Q}_W$ and $\mathscr{Q}_S$. Our agenda is to choose between $\mathscr{Q}_W$ and $\mathscr{Q}_S$ on the basis of monotonicity properties of Q. Before turning to this task, however, let us sidetrack a little bit and look at a different possible way of discrimination.

4.2 Dynamic Notions of Conservativity

Chierchia (1990) claims that there is an *a priori* reason to favor $\mathscr{Q}_W$ over $\mathscr{Q}_S$ regardless of the kind of quantifier Q is. His reason is that $\mathscr{Q}_W$, but not $\mathscr{Q}_S$, can be said to be dynamically conservative. A dynamic generalized quantifier $\mathscr{Q}$ is said to be dynamically conservative in Chierchia's sense if the following is valid:

Dynamic Conservativity 1 (DCONS1).

$$\mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}x(\varphi, \varphi; \psi)$$

DCONS1 is supposed to be the dynamic analogue of CONS. Given that ; is the dynamic counterpart of $\wedge$, there is a good correspondence. It is easy to see that $\mathscr{Q}_W$, but not $\mathscr{Q}_S$, satisfies DCONS1.[20]

[20] There is a proviso that $\text{AQV}(\varphi) \cap \text{FV}(\varphi) = \emptyset$. (See (15) and footnote 13.) This condition is necessary to make many other reasonable statements, e.g., the validity of

Since CONS is a fundamental property of static generalized quantifiers representing natural language determiners, we would naturally expect some dynamic version of CONS to hold of dynamic generalized quantifiers, and DCONS1 is certainly one natural candidate. Chierchia's mistake was to think DCONS1 is the *only* natural dynamic analogue of CONS. He states: 'I see no non-trivial sense in which [strong dynamic determiners, our $\mathscr{Q}_S$—M.K.] may be said to be dynamically conservative' (p. 21). However, there *is* a non-trivial sense in which $\mathscr{Q}_S$ may be said to be dynamically conservative. It is the following:

Dynamic Conservativity 2 (DCONS2).

$$\mathscr{Q}x(\varphi,\psi) \leftrightarrow \mathscr{Q}x(\varphi,\varphi \Rightarrow \psi)$$

It is easy to see that $\mathscr{Q}_S$, but not $\mathscr{Q}_W$, satisfies DCONS2.

Why is DCONS2 a natural dynamic analogue of CONS? Clearly, the static schema corresponding to DCONS2 is

CONS2. $$Qx(\varphi,\psi) \leftrightarrow Qx(\varphi,\varphi \to \psi)$$

But CONS2 is *equivalent* to CONS: they express one and the same condition on Q.[21] Since CONS2 also expresses (static) Conservativity, DCONS2 can also be said to be a natural dynamic version of Conservativity. If so, one cannot choose between $\mathscr{Q}_W$ and $\mathscr{Q}_S$ on the basis of a criterion like conservativity, contrary to Chierchia's claim. Unlike CONS and CONS2, DCONS1 and DCONS2 are not equivalent. They are different, but equally natural, dynamic versions of the notion of conservativity. I do not see any good grounds for choosing between DCONS1 and DCONS2.[22]

$\varphi \Rightarrow \varphi$ or $\varphi;\varphi \simeq \varphi$.

Proof of the claim. As for $\mathscr{Q}_W$, since ; is associative (see (12)), it is sufficient to observe $\varphi \simeq \varphi;\varphi$ if $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$. As for $\mathscr{Q}_S$, observe that $(\varphi \Rightarrow \varphi;\psi) \simeq (\varphi \to \varphi;\psi)$ so that $\mathscr{Q}_S x(\varphi,\varphi;\psi) \leftrightarrow Qx(\varphi,\varphi \to \varphi;\psi) \leftrightarrow Qx(\varphi,\varphi;\psi) \leftrightarrow \mathscr{Q}_W x(\varphi,\psi)$ (assuming conservativity of Q).

[21] *Proof.* Assume CONS. Then

$$\begin{array}{llll} Qx(\varphi,\varphi \to \psi) & \leftrightarrow & Qx(\varphi,\varphi \wedge (\varphi \to \psi)) & \text{(CONS)} \\ & \leftrightarrow & Qx(\varphi,\varphi \wedge \psi) & (\varphi \wedge (\varphi \to \psi) \leftrightarrow \varphi \wedge \psi) \\ & \leftrightarrow & Qx(\varphi,\psi) & \text{(CONS).} \end{array}$$

For the converse, assume CONS2. Then

$$\begin{array}{llll} Qx(\varphi,\varphi \wedge \psi) & \leftrightarrow & Qx(\varphi,\varphi \to \varphi \wedge \psi) & \text{(CONS2)} \\ & \leftrightarrow & Qx(\varphi,\varphi \to \psi) & ((\varphi \to \varphi \wedge \psi) \leftrightarrow (\varphi \to \psi)) \\ & \leftrightarrow & Qx(\varphi,\psi) & \text{(CONS2).} \end{array}$$

(The middle steps make use of EQUI.)

[22] Of course, viewed as *inferential principles*, CONS and CONS2 are not the same. One might argue that DCONS1 is 'more natural' since, like CONS, it corresponds to

We saw earlier that connectives in first-order logic may have more than one dynamic realization in dynamic predicate logic. Here, we are observing a similar phenomenon. Conditions on static generalized quantifiers like Conservativity may have a variety of non-equivalent dynamic counterparts.

In fact, there is yet another notion of dynamic conservativity which holds of both $\mathscr{Q}_W$ and $\mathscr{Q}_S$. This third notion of dynamic conservativity, which I in fact think is the most natural one, can be given in terms of the *-transformation defined in Section 3.

DCONS*. $\quad \mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}x(\varphi, \varphi^* \wedge \psi)$

Recall that φ^* is the result of erasing all potentially active occurrences of dynamic existential quantifiers[23] and

$$\mathrm{ran}([\![\varphi]\!]_{\mathbf{M}}) = [\varphi^*]_{\mathbf{M}},$$

that is, φ^* expresses the 'dynamic effect' of φ. What DCONS* says is this: to know $[\mathscr{Q}x(\varphi, \psi)]_{\mathbf{M}}$, it is not necessary to look at the whole of $[\psi]_{\mathbf{M}}$; knowing $\mathrm{ran}([\![\varphi]\!]_{\mathbf{M}}) \cap [\psi]_{\mathbf{M}}$ is enough. This is a very natural condition, since, like dynamic connectives of DPL, internally dynamic generalized quantifiers are supposed to evaluate the second argument with respect to assignments that are outputs of the first argument.[24]

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a natural form of inference in English:

> Det $\overline{\mathrm{N}}$ VP $\leftrightarrow$ Det $\overline{\mathrm{N}}$ is a $\overline{\mathrm{N}}$ and VP
> E.g.,
> Det farmer who owns a donkey beats it $\leftrightarrow$
> Det farmer who owns a donkey is a farmer who owns a donkey and beats it

Expressing DCONS2 in English is more awkward:

> Det farmer who owns a donkey beats it $\leftrightarrow$
> Det farmer who owns a donkey is such that if he is a farmer who owns a donkey, he beats it

(CONS2 may be expressed as 'Det $\overline{\mathrm{N}}$ VP $\leftrightarrow$ Det $\overline{\mathrm{N}}$ VP if he (she, etc.) is a $\overline{\mathrm{N}}$', but this does not work for DCONS2.) However, unlike Monotonicity, I think Conservativity should be thought of as a principle about interpretation, not inference. It is difficult to imagine English speakers ever resort to paraphrases like the above. Prolix forms like 'Det $\overline{\mathrm{N}}$ is a $\overline{\mathrm{N}}$ and VP' have little use in ordinary discourse; in fact, I even think such paraphrasability may not be immediately recognized.

[23]As before, φ is assumed to be such that it satisfies (15) and (16) in Section 3, that is, $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$ and there is for any variable x at most one potentially active occurrence of $\mathscr{E}x$. Under (15), the second restriction (16) can be avoided by complicating the transformation (see footnote 13).

[24]To capture this 'spirit' of DCONS* without restricting φ to formulas that satisfy (15) and (16), we might introduce * as a new operator in the language, so that $[\varphi^*]_{\mathbf{M}} =$

Three notions of Dynamic Conservativity, when confined to the case $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\psi) = \emptyset$, all turn out to be equivalent and amount to:

$$\mathscr{Q}x(\varphi, \psi) \quad \leftrightarrow \quad \mathscr{Q}x(\varphi, \varphi \wedge \psi) \quad \text{if } \mathrm{AQV}(\varphi) \cap \mathrm{FV}(\psi) = \emptyset.$$

Clearly, dynamic conjunction and dynamic implication satisfy the following, which are analogous to DCONS*:[25]

(41) $$\varphi\,;\psi \quad \leftrightarrow \quad \varphi\,;(\varphi^* \wedge \psi)$$

(42) $$\varphi \Rightarrow \psi \quad \leftrightarrow \quad \varphi \Rightarrow (\varphi^* \wedge \psi)$$

From this it follows that both DCONS1 and DCONS2 imply DCONS*.[26] Therefore, DCONS* is satisfied by both $\mathscr{Q}_W$ and $\mathscr{Q}_S$.

I adopt DCONS* as our official version of dynamic conservativity for the following two reasons. Firstly, DCONS1 and DCONS2 are too strong principles; they seem to do more than what dynamic conservativity should do, since they essentially reduce dynamic generalized quantification to static generalized quantification combined with dynamic conjunction or implication. (On the right-hand side of DCONS1 and DCONS2, no 'dynamic binding' occurs between the two arguments of $\mathscr{Q}$, except in the anomalous

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$\mathrm{ran}([\![\varphi]\!]_{\mathbf{M}})$ for an arbitrary formula φ. In fact, even this can be avoided if we adopt the following form of Dynamic Conservativity:

DCONS. $$\forall x(\varphi \Rightarrow (\psi \leftrightarrow \psi')) \rightarrow (\mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}x(\varphi, \psi')).$$

DCONS implies DCONS* in its operator version, and, when φ satisfies (15) ($\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$), conversely. They both coincide with DCONS* in its transformation version when φ satisfies both (15) and (16). We opt for the transformation version of DCONS* for simplicity.

[25](41) can be turned into a full dynamic equivalence using ; instead of $\wedge$:

$$\varphi\,;\psi \quad \simeq \quad \varphi\,;(\varphi^*;\psi)$$

[26]For, if DCONS1,

$$\begin{array}{lcll} \mathscr{Q}x(\varphi, \varphi^* \wedge \psi) & \leftrightarrow & \mathscr{Q}x(\varphi, \varphi\,;(\varphi^* \wedge \psi)) & \text{(DCONS1)} \\ & \leftrightarrow & \mathscr{Q}x(\varphi, \varphi\,;\psi) & \text{(41)} \\ & \leftrightarrow & \mathscr{Q}x(\varphi, \psi) & \text{(DCONS1)} \end{array}$$

and if DCONS2,

$$\begin{array}{lcll} \mathscr{Q}x(\varphi, \varphi^* \wedge \psi) & \leftrightarrow & \mathscr{Q}x(\varphi, \varphi \Rightarrow \varphi^* \wedge \psi) & \text{(DCONS2)} \\ & \leftrightarrow & \mathscr{Q}x(\varphi, \varphi \Rightarrow \psi) & \text{(42)} \\ & \leftrightarrow & \mathscr{Q}x(\varphi, \psi) & \text{(DCONS2)} \end{array}$$

case $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) \neq \emptyset$.) They restrict the class of dynamic generalized quantifiers too narrowly—for any static generalized quantifier, a unique dynamic generalized quantifier would be determined. In contrast, DCONS* is a principle purely about dynamic quantification, and it leaves many interesting options open. For example, the hypothetical reading of ‘Most farmers who own a donkey beat it’ paraphrased by ‘Most farmers who own a donkey beat most donkeys they own’ can be represented by a dynamic generalized quantifier satisfying DCONS*. Secondly, DCONS* is sufficiently strong to establish some elegant results in the theory of dynamic generalized quantifiers. Stronger notions of dynamic conservativity are not necessary.[27]

4.3 Monotonicity for Dynamic Generalized Quantifiers

A universal property of quantifiers like Conservativity does not decide which of the two schemata (26) and (27) should be used. Our claim is that different quantifiers can choose different schemata, based on their monotonicity properties. To do this, we have to formulate a suitable notion of monotonicity for internally dynamic generalized quantifiers. Recall that in the static case, monotonicity of a quantifier Q is expressed by the following formulas:

Static Monotonicity.

↑MON	$\forall x(\varphi \to \varphi') \to (Qx(\varphi,\psi) \to Qx(\varphi',\psi))$
↓MON	$\forall x(\varphi' \to \varphi) \to (Qx(\varphi,\psi) \to Qx(\varphi',\psi))$
MON↑	$\forall x(\psi \to \psi') \to (Qx(\varphi,\psi) \to Qx(\varphi,\psi'))$
MON↓	$\forall x(\psi' \to \psi) \to (Qx(\varphi,\psi) \to Qx(\varphi,\psi'))$

Suitable dynamic versions of these formulas turn out to be the following. Let $\{y_1,\ldots,y_n\} = \mathrm{AQV}(\varphi)$.

Dynamic Monotonicity.

↑DMON	$\forall x(\varphi \preceq \varphi') \to (\mathscr{Q}x(\varphi,\psi) \to \mathscr{Q}x(\varphi',\psi))$
↓DMON	$\forall x(\varphi' \preceq \varphi) \to (\mathscr{Q}x(\varphi,\psi) \to \mathscr{Q}x(\varphi',\psi))$
DMON↑	$\forall x \forall y_1 \ldots \forall y_n(\psi \to \psi') \to (\mathscr{Q}x(\varphi,\psi) \to \mathscr{Q}x(\varphi,\psi'))$
DMON↓	$\forall x \forall y_1 \ldots \forall y_n(\psi' \to \psi) \to (\mathscr{Q}x(\varphi,\psi) \to \mathscr{Q}x(\varphi,\psi'))$

The reader should recognize an analogy with DEQUI.

In ↑DMON and ↓DMON, the static implication in the premise of ↑MON and ↓MON is replaced by $\preceq$. This is dictated by the fact that, whereas the static denotation of φ exhausts its semantic contribution to $Qx(\varphi,\psi)$, the

[27] I should also mention that DCONS* looks natural from a semantic point of view, if we think of a dynamic generalized quantifier as a family of polyadic quantifiers as outlined earlier. DCONS* would correspond to $\mathscr{Q}^n RS$ iff $\mathscr{Q}^n R(R \cap S)$.

whole dynamic denotation of φ matters to $\mathscr{Q}x(\varphi, \psi)$. Replacing $\rightarrow$ by $\preceq$ effects a shift from static denotation to dynamic denotation.[28] Note that, of the two dynamic connectives in our DPL which correspond to static implication, the 'meta' connective $\preceq$ is the one that makes sense here. This contrasts with the formulation of DCONS2, where $\Rightarrow$ takes the place of $\rightarrow$ in CONS2.

For internally dynamic (externally static) $\mathscr{Q}$, $[\mathscr{Q}x(\varphi, \psi)]_{\mathbf{M}}$ is a function of $[\![\varphi]\!]_{\mathbf{M}}$ and $[\psi]_{\mathbf{M}}$. (ψ does not make a 'dynamic contribution' to $\mathscr{Q}x(\varphi, \psi)$.) Hence, the implication remains static in the premise of DMON$\uparrow$ and DMON$\downarrow$. The extra universal quantifiers $\forall y_1 \dots \forall y_n$ are accounted for by the fact that the free occurrences of $y_1, \dots, y_n$ in ψ become bound in $\mathscr{Q}x(\varphi, \psi)$.[29]

A dynamic generalized quantifier $\mathscr{Q}$ is said to be dynamically upward monotone in the first argument (etc.) if it validates the $\uparrow$DMON formula (etc.). We say $\mathscr{Q}$ is $\uparrow$DMON (etc.), for short.

This dynamic notion of Monotonicity is adequate to capture monotonicity inference in donkey sentences. Let us consider the earlier non-inference from *No man who owns a house sprinkles it* to *No man who owns a garden sprinkles it*, in models where every man who owns a garden owns a house. Here, we have

$$\begin{aligned}&\forall x(\mathrm{man}(x)\,;\mathscr{E}y(\mathrm{garden}(y)\,;\mathrm{own}(x,y))\\&\qquad \rightarrow \mathrm{man}(x)\,;\mathscr{E}y(\mathrm{house}(y)\,;\mathrm{own}(x,y))),\end{aligned}$$

but the static implication $\rightarrow$ cannot be replaced by the dynamic $\preceq$.[30] Therefore, we cannot expect to derive

$$\mathscr{N}\mathscr{O}x(\mathrm{man}(x)\,;\mathscr{E}y(\mathrm{garden}(y)\,;\mathrm{own}(x,y)),\,\mathrm{sprinkle}(x,y))$$

[28] In the simple case where $\mathrm{AQV}(\varphi) = \mathrm{AQV}(\varphi') = \{y_1, \dots, y_n\}$, we have

$$\forall x(\varphi \preceq \varphi') \quad \leftrightarrow \quad \forall x \forall y_1 \dots y_n(\varphi^* \rightarrow \varphi'^*).$$

In the general case, the relation between the two formulas is complex.

[29] If DCONS* is assumed, reference to $\mathrm{AQV}(\varphi)$ in DMON$\uparrow$ and DMON$\downarrow$ can be avoided by adopting the following:

$$\begin{array}{ll}\mathrm{DMON}\uparrow & \forall x(\varphi \Rightarrow (\psi \rightarrow \psi')) \rightarrow (\mathscr{Q}x(\varphi, \psi) \rightarrow \mathscr{Q}x(\varphi, \psi'))\\ \mathrm{DMON}\downarrow & \forall x(\varphi \Rightarrow (\psi' \rightarrow \psi)) \rightarrow (\mathscr{Q}x(\varphi, \psi) \rightarrow \mathscr{Q}x(\varphi, \psi'))\end{array}$$

The above version implies DCONS* (or DCONS of footnote 24).

[30] Note that

$$\forall x(\mathrm{man}(x)\,;\mathscr{E}y(\mathrm{garden}(y)\,;\mathrm{own}(x,y)) \preceq \mathrm{man}(x)\,;\mathscr{E}y(\mathrm{house}(y)\,;\mathrm{own}(x,y)))$$

is equivalent to

$$\forall x \forall y(\mathrm{man}(x) \wedge \mathrm{garden}(y) \wedge \mathrm{own}(x,y) \rightarrow \mathrm{man}(x) \wedge \mathrm{house}(y) \wedge \mathrm{own}(x,y)).$$

See footnote 28.

from

$$\mathscr{NO}x(\text{man}(x)\,;\mathscr{E}y(\text{house}(y)\,;\text{own}(x,y)),\,\text{sprinkle}(x,y)),$$

even if $\mathscr{NO}$ is a dynamic generalized quantifier that is $\downarrow$DMON. In contrast, in the case of valid inference from *No farmer who owns a donkey beats it* to *No farmer who owns a female donkey beats it*, the necessary premise does hold:

$$\begin{aligned}\forall x(&\text{farmer}(x)\,;\mathscr{E}y(\text{female}(y)\,;\text{donkey}(y)\,;\text{own}(x,y))\\ &\preceq \text{farmer}(x)\,;\mathscr{E}y(\text{donkey}(y)\,;\text{own}(x,y))).\end{aligned}$$

Dynamic left monotonicity of $\mathscr{Q}_W$ and $\mathscr{Q}_S$. Given a double monotone static quantifier Q, we can show either $\mathscr{Q}_W$ or $\mathscr{Q}_S$, but not both, turns out to be dynamically left monotone. First, note the following monotonicity behavior of ; and $\Rightarrow$. Let $\{y_1,\ldots,y_n\} = \text{AQV}(\varphi)$.

(43) $$(\varphi \preceq \varphi') \quad\rightarrow\quad (\varphi\,;\psi \rightarrow \varphi'\,;\psi)$$

(44) $$\forall y_1 \ldots \forall y_n(\psi \rightarrow \psi') \quad\rightarrow\quad (\varphi\,;\psi \rightarrow \varphi\,;\psi')$$

(45) $$(\varphi' \preceq \varphi) \quad\rightarrow\quad ((\varphi \Rightarrow \psi) \rightarrow (\varphi' \Rightarrow \psi))$$

(46) $$\forall y_1 \ldots \forall y_n(\psi \rightarrow \psi') \quad\rightarrow\quad ((\varphi \Rightarrow \psi) \rightarrow (\varphi \Rightarrow \psi'))$$

For example, let Q be $\downarrow$MON$\uparrow$. Since

$$\forall x(\varphi' \preceq \varphi) \quad\rightarrow\quad \forall x(\varphi' \rightarrow \varphi)$$

and by (45),

$$\forall x(\varphi' \preceq \varphi) \quad\rightarrow\quad \forall x((\varphi \Rightarrow \psi) \rightarrow (\varphi' \Rightarrow \psi)),$$

we get

$$\forall x(\varphi' \preceq \varphi) \quad\rightarrow\quad (Qx(\varphi, \varphi \Rightarrow \psi) \rightarrow Qx(\varphi', \varphi' \Rightarrow \psi))$$

by $\downarrow$MON and MON$\uparrow$. This means that $\mathscr{Q}_S$ is $\downarrow$DMON.

Table 4 shows the dynamic monotonicity properties of $\mathscr{Q}_W$ and $\mathscr{Q}_S$ for double monotone Q.[31]

Left Monotonicity Principle. We are now in a position to be able to formulate a principle about interpretations of donkey sentences that explains the correlation given in Table 1 precisely in terms of dynamic predicate logic with generalized quantifiers. The interpretation of a donkey sentence of the form

$$\text{Det}\ \ \overline{\text{N}}\ \ \text{VP},$$

[31] $\not\uparrow$DMON ($\not\downarrow$DMON) here means that the $\uparrow$DMON ($\downarrow$DMON) formula is not in general valid.

Q	$\mathscr{Q}_W$	$\mathscr{Q}_S$
$\uparrow$MON$\uparrow$	$\uparrow$DMON$\uparrow$	$\not\uparrow$DMON$\uparrow$
$\uparrow$MON$\downarrow$	$\not\uparrow$DMON$\downarrow$	$\uparrow$DMON$\downarrow$
$\downarrow$MON$\uparrow$	$\not\downarrow$DMON$\uparrow$	$\downarrow$DMON$\uparrow$
$\downarrow$MON$\downarrow$	$\downarrow$DMON$\downarrow$	$\not\downarrow$DMON$\downarrow$

Table 4: Dynamic monotonicity of $\mathscr{Q}_W$ and $\mathscr{Q}_S$.

where Det 'means' Q, is assumed to be represented by a formula of DPL$(Q, \mathscr{Q})$

$$\mathscr{Q}x(\varphi, \psi),$$

where $\mathscr{Q}$ is either $\mathscr{Q}_W$ or $\mathscr{Q}_S$, and φ and ψ are the translations of $\overline{\mathrm{N}}$ and VP, respectively, in DPL. (Assume that indefinite noun phrases are translated using $\mathscr{E}$.) The principle is the following:

(47) Left Monotonicity Principle. If Q is $\uparrow$MON ($\downarrow$MON), the suitable dynamic version $\mathscr{Q}$ of Q should be $\uparrow$DMON ($\downarrow$DMON).

By Table 4, this correctly explains the correlation given in Table 1. The Left Monotonicity Principle ensures that the interpretation of a donkey sentence with a left monotone determiner validates monotonicity inference like (3) and (4).

Note that right monotonicity is preserved by both $\mathscr{Q}_W$ and $\mathscr{Q}_S$. Hence we can also say:

(48) Monotonicity Principle. The suitable dynamic version $\mathscr{Q}$ of Q should preserve the monotonicity properties of Q (as dynamic monotonicity).

4.4 Double Monotonicity Characterization of Dynamic Generalized Quantifiers

We have shown that dynamic left monotonicity can be used to discriminate between $\mathscr{Q}_W$ and $\mathscr{Q}_S$. In fact, we do not have to restrict our attention to $\mathscr{Q}_W$ and $\mathscr{Q}_S$ from the start. Under minimal assumptions, it can be shown that dynamic monotonicity in both arguments *uniquely determines* a dynamic counterpart $\mathscr{Q}$ of a given static generalized quantifier Q. Of course, such a $\mathscr{Q}$ must be either $\mathscr{Q}_W$ or $\mathscr{Q}_S$.

An obvious condition to impose on the relationship between Q and $\mathscr{Q}$ is the following, which we call Agreement.

Agreement. If $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\psi) = \emptyset$,

$$\mathscr{Q}x(\varphi, \psi) \leftrightarrow Qx(\varphi, \psi)$$

It says that dynamic quantification should reduce to static quantification when there is no 'dynamic binding' involved. Recall that analogous situations obtained for ; and $\Rightarrow$ ((19) and (20)). Obviously, $\mathscr{Q}_W$ and $\mathscr{Q}_S$ both satisfy this condition if Q satisfies CONS.

Agreement looks like a very innocuous principle; dynamic $\mathscr{Q}$ should certainly be a 'conservative extension' of static Q. Nevertheless, combined with other natural principles, it implies rather strong consequences. In the following, DEQUI is always assumed. (See Section 4.6 for proof of the results of this section.)

LEMMA 1. If $\mathscr{Q}$ satisfies DCONS* and Agreement, then the following holds:

$$\begin{aligned}&\forall x((\varphi \Rightarrow \psi) \vee \neg(\varphi\,;\psi)) \rightarrow \\ &\qquad (\mathscr{Q}x(\varphi,\psi) \leftrightarrow \mathscr{Q}_W x(\varphi,\psi)) \wedge (\mathscr{Q}x(\varphi,\psi) \leftrightarrow \mathscr{Q}_S x(\varphi,\psi))\end{aligned}$$

Lemma 1 corresponds to the fact that, with respect to sentences like *Most farmers who own a donkey beat it*, 'people have firm intuitions about situations where farmers are consistent about their donkey-beating' (Rooth 1987), that is, when each donkey-owning farmer beats either all of their donkeys or none of them. In such situations, both the weak reading and the strong reading—which become equivalent—adequately capture the intuitions.

As a special case of Lemma 1, we have

COROLLARY 1. Under the same conditions, the following holds:

$$\begin{aligned}&\forall x \exists z_1 \ldots \exists z_n(\varphi \Rightarrow y_1 = z_1 \wedge \cdots \wedge y_n = z_n) \rightarrow \\ &\quad (\mathscr{Q}x(\varphi,\psi) \leftrightarrow Qx(\varphi, \exists z_1 \ldots \exists z_n((\varphi \Rightarrow y_1 = z_1 \wedge \cdots \wedge y_n = z_n) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \wedge\, \psi[z_1/y_1,\ldots,z_n/y_n])))\end{aligned}$$

Here, $\{y_1,\ldots,y_n\} = \mathrm{AQV}(\varphi) \cap \mathrm{FV}(\psi)$, $z_1,\ldots,z_n$ are new variables, and $\psi[z_1/y_1,\ldots,z_n/y_n]$ is the result of replacing all free occurrences of y_i in ψ by z_i ($1 \leq i \leq n$).

Corollary 1 says that if the value for the 'donkey variable' is unique per value for the 'farmer variable', then the donkey variable can be replaced by an appropriate definite description, and the quantification can be taken to be static. Notice that the antecedent expresses the uniqueness condition, and the quantification in the second argument of Q amounts to a Russellian treatment of definite description. This corresponds to the empirical fact that when the 'uniqueness presupposition' of the donkey pronoun is met, then there is no question whatsoever about the truth conditions of the donkey sentence, and the paraphrase with a definite description is entirely adequate. It is interesting that this automatically follows from Agreement and DCONS*.

Now with Agreement and DCONS*, dynamic double monotonicity is sufficient to pin down $\mathscr{Q}$ uniquely from Q.

PROPOSITION 2. Assume that $\mathscr{Q}$ satisfies Agreement and DCONS*. If $\mathscr{Q}$ is moreover DMON↑,

$$\mathscr{Q}_S x(\varphi,\psi) \quad\rightarrow\quad \mathscr{Q}x(\varphi,\psi) \quad\rightarrow\quad \mathscr{Q}_W x(\varphi,\psi).$$

If $\mathscr{Q}$ is DMON↓, the reverse implications hold.

PROPOSITION 3. Assume that $\mathscr{Q}$ satisfies Agreement and DCONS*. Then if $\mathscr{Q}$ is ↑DMON,

$$\mathscr{Q}_W x(\varphi,\psi) \quad\rightarrow\quad \mathscr{Q}x(\varphi,\psi) \quad \text{and} \quad \mathscr{Q}_S x(\varphi,\psi) \quad\rightarrow\quad \mathscr{Q}x(\varphi,\psi).$$

If $\mathscr{Q}$ is ↓DMON, the reverse implications hold.

Proposition 3 uses Lemma 1. By the two Propositions, if Q is ↑MON↑, $\mathscr{Q}_W$ is the only ↑DMON↑ $\mathscr{Q}$ which satisfies Agreement and DCONS*. Likewise for the other three double monotonicity patterns. In this way, dynamic double monotonicity uniquely determines a dynamic generalized quantifier corresponding to a given static one.

The significance of the results in this section can be summarized as follows. Rather than thinking of $\mathscr{Q}_W$ and $\mathscr{Q}_S$ as two possible options to choose from, we may think that no concrete choice for a dynamic counterpart $\mathscr{Q}$ of a static Q is given in advance, but that there are minimal conditions to be satisfied by any possible dynamic counterpart of a static Q, namely Agreement and DCONS*.[32] Imposing dynamic double monotonicity as a further requirement then amounts to an *implicit definition* of $\mathscr{Q}$ out of Q.

The consequence of this to the semantics of donkey sentences is that we need not think that the weak and strong readings are the two possible interpretations of donkey sentences that would be allowed in principle, of which one or the other is picked by the Left Monotonicity Principle when the determiner is double monotone. Instead, we may think that in the absence of information about specific properties of the determiner, the grammar would not assign any concrete interpretation to a donkey sentence at all. The Monotonicity Principle (48) can then be considered as a principle that *forms* a concrete interpretation, fleshing out a schematic, partially specified meaning provided by the grammar.[33]

[32] Another natural condition is a suitable dynamic version of EXT. See Section 4.5.
[33] More discussion on this point will be found in Kanazawa 1993.

4.5 Monotonicity and Preservation

In Section 2, we mentioned an equivalence between left monotonicity on the one hand and preservation under extensions/submodels of a sentence of a certain form on the other. It turns out that this can be extended to the dynamic setting. Firstly, in first-order logic with generalized quantifiers, one half of Proposition 1 can be strengthened:

PROPOSITION 4. Assume that Q has EXT and CONS. Then Q is $\uparrow$MON ($\downarrow$MON) if and only if every sentence of the form $Qx(\varphi,\psi)$, where φ is existential and ψ is quantifier-free, is preserved under extensions (submodels).[34]

Now let us define a dynamic notion of Extension as follows:

Dynamic Extension (DEXT). For any $\mathbf{M}$, $\mathbf{N}$, $s\colon VAR \to M$, and $s'\colon VAR \to N$, if

$$\{\,\langle a,b_1,\ldots,b_n\rangle \in M^{n+1} \mid s(a/x)\,[\![\varphi]\!]_{\mathbf{M}}\, s(a/x,b_1/y_1,\ldots,b_n/y_n)\,\}$$
$$= \{\,\langle a,b_1,\ldots,b_n\rangle \in N^{n+1} \mid s'(a/x)\,[\![\varphi]\!]_{\mathbf{N}}\, s'(a/x,b_1/y_1,\ldots,b_n/y_n)\,\}$$

and

$$\{\,\langle a,b_1,\ldots,b_n\rangle \in M^{n+1} \mid \mathbf{M} \models \psi[s(a/x,b_1/y_1,\ldots,b_n/y_n)]\,\}$$
$$= \{\,\langle a,b_1,\ldots,b_n\rangle \in N^{n+1} \mid \mathbf{N} \models \psi[s'(a/x,b_1/y_1,\ldots,b_n/y_n)]\,\},$$

where $\{y_1,\ldots,y_n\} = \mathrm{AQV}(\varphi)$,[35] then

$$\mathbf{M} \models \mathcal{Q}x(\varphi,\psi)[s] \quad \text{iff} \quad \mathbf{N} \models \mathcal{Q}x(\varphi,\psi)[s'].$$

Then we have a straightforward dynamic version of Proposition 4:

PROPOSITION 5. Assume that $\mathcal{Q}$ satisfies DEXT and DCONS*. Then $\mathcal{Q}$ is $\uparrow$DMON ($\downarrow$DMON) if and only if every sentence of the form $\mathcal{Q}x(\varphi,\psi)$, where φ is existential and ψ is quantifier-free, is preserved under extensions (submodels).

A DPL formula φ is *existential* if it is of the form $\mathcal{E}x_1\ldots\mathcal{E}x_n\exists x_{n+1}\ldots\exists x_{n+m}\psi$ for a quantifier-free ψ. (See Section 4.6 for proof of Proposition 5.) Proposition 5 may be seen as an indication that our DEXT, DCONS*, and DMON are natural dynamic analogues of the corresponding static conditions.

[34] A first-order formula φ is called existential if it is of the form $\exists x_1\ldots x_n\psi$ for a quantifier-free ψ.

[35] As usual, $x \notin \mathrm{AQV}(\varphi)$ is assumed.

The fact that a sentence is preserved under extensions or submodels means that the truth of the sentence in one model may be inferred from the truth of the same sentence in another model. Such inference can be regarded as a model-theoretic mode of monotonicity inference. The relevance of this to the semantics of donkey sentences is discussed at length in Kanazawa 1993.

4.6 Proofs

In what follows, φ stands for a formula of dynamic predicate logic with generalized quantifiers such that $\mathrm{AQV}(\varphi) \cap \mathrm{FV}(\varphi) = \emptyset$ and there is at most one potentially active occurrence of $\mathscr{E}x$ in φ for any variable x. ψ stands for an arbitrary formula. Let $\{y_1, \ldots, y_n\} = \mathrm{AQV}(\varphi)$. φ^* is the result of erasing all potentially active occurrences of dynamic existential quantifiers in φ. Note the following equivalences:

$$\begin{array}{rcl}
\varphi & \simeq & \mathscr{E}y_1 \ldots \mathscr{E}y_n \varphi^* \\
& \leftrightarrow & \exists y_1 \ldots \exists y_n \varphi^* \\
\varphi ; \psi & \simeq & \mathscr{E}y_1 \ldots \mathscr{E}y_n (\varphi^* ; \psi) \\
& \leftrightarrow & \exists y_1 \ldots \exists y_n (\varphi^* \wedge \psi) \\
\varphi \Rightarrow \psi & \simeq & \forall y_1 \ldots \forall y_n (\varphi^* \rightarrow \psi)
\end{array}$$

LEMMA 1. If $\mathscr{Q}$ satisfies DCONS* and Agreement, then the following holds:

$$\begin{array}{l}
\forall x((\varphi \Rightarrow \psi) \vee \neg(\varphi ; \psi)) \rightarrow \\
\qquad (\mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}_W x(\varphi, \psi)) \wedge (\mathscr{Q}x(\varphi, \psi) \leftrightarrow \mathscr{Q}_S x(\varphi, \psi))
\end{array}$$

Proof. By DCONS*,

$$\mathscr{Q}x(\varphi, \psi) \quad \leftrightarrow \quad \mathscr{Q}x(\varphi, \varphi^* \wedge \psi)$$

and by Agreement and DCONS*,

$$\begin{array}{rcl}
\mathscr{Q}_W x(\varphi, \psi) & \leftrightarrow & Qx(\varphi, \varphi ; \psi) \\
& \leftrightarrow & \mathscr{Q}x(\varphi, \varphi ; \psi) \\
& \leftrightarrow & \mathscr{Q}x(\varphi, \varphi^* \wedge (\varphi ; \psi)) \\
\mathscr{Q}_S x(\varphi, \psi) & \leftrightarrow & Qx(\varphi, \varphi \Rightarrow \psi) \\
& \leftrightarrow & \mathscr{Q}x(\varphi, \varphi \Rightarrow \psi) \\
& \leftrightarrow & \mathscr{Q}x(\varphi, \varphi^* \wedge (\varphi \Rightarrow \psi)).
\end{array}$$

Therefore, it suffices to show that

$$\forall x((\varphi \Rightarrow \psi) \vee \neg(\varphi ; \psi))$$

implies

$$\forall x \forall y_1 \ldots \forall y_n (\varphi^* \wedge \psi \leftrightarrow \varphi^* \wedge (\varphi ; \psi))$$

and

$$\forall x \forall y_1 \ldots \forall y_n (\varphi^* \wedge \psi \leftrightarrow \varphi^* \wedge (\varphi \Rightarrow \psi)),$$

which, by DEQUI, imply

$$\mathscr{Q}x(\varphi, \varphi^* \wedge \psi) \quad \leftrightarrow \quad \mathscr{Q}x(\varphi, \varphi^* \wedge (\varphi ; \psi))$$

and

$$\mathscr{Q}x(\varphi, \varphi^* \wedge \psi) \quad \leftrightarrow \quad \mathscr{Q}x(\varphi, \varphi^* \wedge (\varphi \Rightarrow \psi)).$$

Notice that

$$\varphi^* \wedge \psi \rightarrow \varphi^* \wedge (\varphi ; \psi)$$

and

$$\varphi^* \wedge (\varphi \Rightarrow \psi) \rightarrow \varphi^* \wedge \psi$$

are valid. It remains to show that

$$\varphi^* \wedge (\varphi ; \psi) \quad \rightarrow \quad \varphi^* \wedge \psi \tag{1}$$

$$\varphi^* \wedge \psi \quad \rightarrow \quad \varphi^* \wedge (\varphi \Rightarrow \psi) \tag{2}$$

follow from

$$(\varphi \Rightarrow \psi) \vee \neg(\varphi ; \psi). \tag{3}$$

As for (1), if $(\varphi \Rightarrow \psi)$, then φ^* implies $\varphi^* \wedge \psi$ and if $\neg(\varphi ; \psi)$, then the antecedent of (1) is false. So (1) holds under (3). As for (2), if $(\varphi \Rightarrow \psi)$, φ^* implies the consequent of (2), and if $\neg(\varphi ; \psi)$, the antecedent of (2) is false. So (2) holds under (3).

PROPOSITION 2. Assume that $\mathscr{Q}$ satisfies Agreement and DCONS*. If $\mathscr{Q}$ is moreover DMON↑,

$$\mathscr{Q}_S x(\varphi, \psi) \quad \rightarrow \quad \mathscr{Q}x(\varphi, \psi) \quad \rightarrow \quad \mathscr{Q}_W x(\varphi, \psi).$$

If $\mathscr{Q}$ is DMON↓, the reverse implications hold.

Proof.

$$\begin{array}{llll}
\mathscr{Q}_S x(\varphi, \psi) & \leftrightarrow & Qx(\varphi, \varphi \Rightarrow \psi) & \text{(Definition of } \mathscr{Q}_S\text{)} \\
 & \leftrightarrow & \mathscr{Q}x(\varphi, \varphi \Rightarrow \psi) & \text{(Agreement)} \\
 & \rightarrow & \mathscr{Q}x(\varphi, \varphi^* \rightarrow \psi) & \text{(DMON}\uparrow\text{)} \\
 & \leftrightarrow & \mathscr{Q}x(\varphi, \psi) & \text{(DCONS*)}
\end{array}$$

(Note $\forall x \forall y_1 \dots \forall y_n((\varphi \Rightarrow \psi) \to (\varphi^* \to \psi))$.)

$$\begin{array}{rcll} \mathcal{Q}x(\varphi,\psi) & \leftrightarrow & \mathcal{Q}x(\varphi,\varphi^* \wedge \psi) & \text{(DCONS*)} \\ & \to & \mathcal{Q}x(\varphi,\varphi\,;\psi) & \text{(DMON}\uparrow\text{)} \\ & \leftrightarrow & Qx(\varphi,\varphi\,;\psi) & \text{(Agreement)} \\ & \leftrightarrow & \mathcal{Q}_W x(\varphi,\psi) & \text{(Definition of } \mathcal{Q}_W\text{)} \end{array}$$

(Note $\forall x \forall y_1 \dots \forall y_n((\varphi^* \wedge \psi) \to (\varphi\,;\psi))$.)

The DMON$\downarrow$ case is similar.

PROPOSITION 3. Assume that $\mathcal{Q}$ satisfies Agreement and DCONS*. Then if $\mathcal{Q}$ is $\uparrow$DMON,

$$\mathcal{Q}_W x(\varphi,\psi) \quad\to\quad \mathcal{Q}x(\varphi,\psi) \quad \text{and} \quad \mathcal{Q}_S x(\varphi,\psi) \quad\to\quad \mathcal{Q}x(\varphi,\psi)$$

If $\mathcal{Q}$ is $\downarrow$DMON, the reverse implications hold.

Proof. The crux of the proof consists in finding formulas χ and σ that satisfy the following conditions:

$$\begin{array}{llll} (1) & \forall x(\chi \preceq \varphi) & (5) & \forall x(\sigma \preceq \varphi) \\ (2) & \forall x(\chi \leftrightarrow \varphi) & (6) & \forall x(\sigma \leftrightarrow \varphi) \\ (3) & \forall x(\chi\,;\psi \leftrightarrow \varphi\,;\psi) & (7) & \forall x(\sigma \Rightarrow \psi \leftrightarrow \varphi \Rightarrow \psi) \\ (4) & \forall x((\chi \Rightarrow \psi) \vee \neg(\chi\,;\psi)) & (8) & \forall x((\sigma \Rightarrow \psi) \vee \neg(\sigma\,;\psi)) \end{array}$$

If such χ and σ are found,

$$\begin{array}{rcll} \mathcal{Q}_W x(\varphi,\psi) & \leftrightarrow & Qx(\varphi,\varphi\,;\psi) & \text{(Definition of } \mathcal{Q}_W\text{)} \\ & \leftrightarrow & Qx(\chi,\chi\,;\psi) & \text{by (2), (3), and EQUI} \\ & \leftrightarrow & \mathcal{Q}_W x(\chi,\psi) & \text{(Definition of } \mathcal{Q}_W\text{)} \\ & \leftrightarrow & \mathcal{Q}x(\chi,\psi) & \text{by (4) and Lemma 1} \\ & \to & \mathcal{Q}x(\varphi,\psi) & \text{by (1) and } \uparrow\text{DMON} \end{array}$$

and

$$\begin{array}{rcll} \mathcal{Q}_S x(\varphi,\psi) & \leftrightarrow & Qx(\varphi,\varphi \Rightarrow \psi) & \text{(Definition of } \mathcal{Q}_S\text{)} \\ & \leftrightarrow & Qx(\sigma,\sigma \Rightarrow \psi) & \text{by (6), (7), and EQUI} \\ & \leftrightarrow & \mathcal{Q}_S x(\sigma,\psi) & \text{(Definition of } \mathcal{Q}_S\text{)} \\ & \leftrightarrow & \mathcal{Q}x(\sigma,\psi) & \text{by (8) and Lemma 1} \\ & \to & \mathcal{Q}x(\varphi,\psi) & \text{by (5) and } \uparrow\text{DMON.} \end{array}$$

(If $\mathcal{Q}$ is $\downarrow$DMON, the last implication is reversed.) Let

$$\chi^* = (\varphi^* \wedge \psi) \vee (\neg(\varphi\,;\psi) \wedge \varphi^*)$$

and

$$\sigma^* = (\varphi^* \wedge \neg\psi) \vee ((\varphi \Rightarrow \psi) \wedge \varphi^*).$$

Then $\chi = \mathscr{E}y_1 \dots \mathscr{E}y_n\chi^*$ and $\sigma = \mathscr{E}y_1 \dots \mathscr{E}y_n\sigma^*$ are the desired formulas. We leave the verification of (1)–(8) to the reader.

PROPOSITION 5. Assume that $\mathscr{Q}$ satisfies DEXT and DCONS*. Then $\mathscr{Q}$ is $\uparrow$DMON ($\downarrow$DMON) if and only if every sentence of the form $\mathscr{Q}x(\varphi, \psi)$, where φ is existential and ψ is quantifier-free, is preserved under extensions (submodels).

Proof.
Only if. Suppose $\mathscr{Q}$ is $\downarrow$DMON (the $\uparrow$DMON case is similar), and let

$$\varphi = \mathscr{E}y_1 \dots \mathscr{E}y_n \exists z_1 \dots \exists z_m \chi$$

where χ is a quantifier-free formula with free variables $x, y_1, \dots, y_n, z_1, \dots, z_m$. Let ψ be a quantifier-free formula with free variables $x, y_1, \dots, y_n$. Since dynamic connectives are equivalent to corresponding static connectives when they combine quantifier-free formulas, $\varphi^* = \exists z_1 \dots \exists z_m \chi$ is equivalent to a first-order existential formula, and ψ is equivalent to a first-order quantifier-free formula. Let $\mathbf{N} \subseteq \mathbf{M}$. Then, as in first-order logic, for all $a, b_1, \dots, b_n \in N$,

$$\mathbf{N} \models \varphi^*[a, b_1, \dots, b_n] \quad \text{implies} \quad \mathbf{M} \models \varphi^*[a, b_1, \dots, b_n]$$

and

$$\mathbf{N} \models \psi[a, b_1, \dots, b_n] \quad \text{if and only if} \quad \mathbf{M} \models \psi[a, b_1, \dots, b_n].$$

Expand the language by adding a new unary predicate P, and expand $\mathbf{M}$ and $\mathbf{N}$ to $\mathbf{M}'$ and $\mathbf{N}'$ by putting $P^{\mathbf{M}'} = P^{\mathbf{N}'} = N$ ($\mathbf{M}'$ and $\mathbf{N}'$ are otherwise the same as $\mathbf{M}$ and $\mathbf{N}$). Note that $\mathbf{N}' \subseteq \mathbf{M}'$. Let

$$\varphi' = \mathscr{E}y_1 \dots \mathscr{E}y_n \exists z_1 \dots \exists z_m (Px \wedge Py_1 \wedge \dots \wedge Py_n \wedge Pz_1 \wedge \dots \wedge Pz_m \wedge \chi)$$

and

$$\psi' = Px \wedge Py_1 \wedge \dots \wedge Py_n \wedge Pz_1 \wedge \dots \wedge Pz_m \wedge \psi.$$

Then for all $a, b_1, \dots, b_n \in M$,

$$\begin{aligned}
\mathbf{M}' \models \varphi'^*[a, b_1, \dots, b_n] \quad &\text{iff} \quad \mathbf{N} \models \varphi^*[a, b_1, \dots, b_n] \\
&\text{iff} \quad \mathbf{N}' \models \varphi'^*[a, b_1, \dots, b_n] \\
\mathbf{M}' \models \psi'[a, b_1, \dots, b_n] \quad &\text{iff} \quad \mathbf{N} \models \psi[a, b_1, \dots, b_n] \\
&\text{iff} \quad \mathbf{N}' \models \psi'[a, b_1, \dots, b_n].
\end{aligned}$$

This implies that the preconditions of DEXT hold for $\mathscr{Q}x(\varphi', \psi')$ (and for an arbitrary $s{:}\, VAR \to N$). So

$$\mathbf{M}' \models \mathscr{Q}x(\varphi', \psi') \quad \text{iff} \quad \mathbf{N}' \models \mathscr{Q}x(\varphi', \psi').$$

Clearly,

$$\mathbf{M}' \models \forall x(\varphi' \preceq \varphi)$$
$$\mathbf{M}' \models \forall x \forall y_1 \ldots \forall y_n(\varphi'^* \wedge \psi \leftrightarrow \varphi'^* \wedge \psi').$$

Therefore,

$$\begin{array}{rcll} \mathbf{M} \models \mathscr{Q}x(\varphi, \psi) & \leftrightarrow & \mathbf{M}' \models \mathscr{Q}x(\varphi, \psi) & \\ & \rightarrow & \mathbf{M}' \models \mathscr{Q}x(\varphi', \psi) & (\downarrow\text{DMON}) \\ & \leftrightarrow & \mathbf{M}' \models \mathscr{Q}x(\varphi', \psi') & (\text{DCONS*, DEQUI}) \\ & \leftrightarrow & \mathbf{N}' \models \mathscr{Q}x(\varphi', \psi') & (\text{DEXT}) \\ & \leftrightarrow & \mathbf{N}' \models \mathscr{Q}x(\varphi, \psi) & (\text{DEQUI}) \\ & \leftrightarrow & \mathbf{N} \models \mathscr{Q}x(\varphi, \psi). & \end{array}$$

If. Suppose that any sentence of the form $\mathscr{Q}x(\varphi, \psi)$, where φ is existential and ψ quantifier-free, is preserved under extensions. (The submodel case is similar.) Let

$$\mathbf{M} \models \forall x(\varphi \preceq \varphi')[s].$$

We treat the simple case where $x \notin \text{AQV}(\varphi) = \text{AQV}(\varphi') = \{y_1, \ldots, y_n\}$ and $\text{FV}(\varphi) = \text{FV}(\varphi') = \{x\}$. (It is possible to reduce the general case to this special case by using Subordinate Renaming and a trick of expanding the model by adding new constants.) Take the language having four predicate symbols R, S, T, and U, and let $\mathbf{M}_1 = \langle M, R^{\mathbf{M}_1}, S^{\mathbf{M}_1}, T^{\mathbf{M}_1}, U^{\mathbf{M}_1} \rangle$, where

$$\begin{aligned} R^{\mathbf{M}_1} &= \{ \langle a, b_1, \ldots, b_n \rangle \in M^{n+1} \mid s(a/x)[\![\varphi]\!]_{\mathbf{M}} s(a/x, b_1/y_1, \ldots, b_n/y_n) \}, \\ S^{\mathbf{M}_1} &= \{ \langle a, b_1, \ldots, b_n \rangle \in M^{n+1} \mid s(a/x)[\![\varphi']\!]_{\mathbf{M}} s(a/x, b_1/y_1, \ldots, b_n/y_n) \}, \\ T^{\mathbf{M}_1} &= \emptyset, \\ U^{\mathbf{M}_1} &= \{ \langle a, b_1, \ldots, b_n \rangle \in M^{n+1} \mid \mathbf{M} \models \psi[s(a/x, b_1/y_1, \ldots, b_n/y_n)] \}. \end{aligned}$$

We have $R^{\mathbf{M}_1} \subseteq S^{\mathbf{M}_1}$. Let $\mathbf{N}_1 = \langle M \cup \{c\}, R^{\mathbf{N}_1}, S^{\mathbf{N}_1}, T^{\mathbf{N}_1}, U^{\mathbf{N}_1} \rangle$ ($c \notin M$), where

$$\begin{array}{rcl} R^{\mathbf{N}_1} & = & R^{\mathbf{M}_1}, \\ S^{\mathbf{N}_1} & = & S^{\mathbf{M}_1}, \\ T^{\mathbf{N}_1} & = & \{c\}, \\ U^{\mathbf{N}_1} & = & U^{\mathbf{M}_1}. \end{array}$$

We have $\mathbf{M}_1 \subseteq \mathbf{N}_1$. Let

$$\chi = \mathscr{E} y_1 \ldots \mathscr{E} y_n \exists z(R(x, y_1, \ldots, y_n) \vee (S(x, y_1, \ldots, y_n) \wedge T(z))).$$

χ is an existential formula. Clearly,

$$\{ \langle a, b_1, \ldots, b_n \rangle \in M^{n+1} \mid s(a/x)[\![\chi]\!]_{\mathbf{M}_1} s(a/x, b_1/y_1, \ldots, b_n/y_n) \} = R^{\mathbf{M}_1},$$
$$\{ \langle a, b_1, \ldots, b_n \rangle \in N^{n+1} \mid s(a/x)[\![\chi]\!]_{\mathbf{N}_1} s(a/x, b_1/y_1, \ldots, b_n/y_n) \} = S^{\mathbf{M}_1}.$$

Therefore,

$$\begin{array}{rcll}
\mathbf{M} \models \mathscr{Q}x(\varphi,\psi)[s] & \leftrightarrow & \mathbf{M}_1 \models \mathscr{Q}x(\chi, U(x, y_1, \ldots, y_n))[s] & \text{(DEXT)} \\
& \leftrightarrow & \mathbf{M}_1 \models \mathscr{Q}x(\chi, U(x, y_1, \ldots, y_n)) & \\
& \rightarrow & \mathbf{N}_1 \models \mathscr{Q}x(\chi, U(x, y_1, \ldots, y_n)) \text{ by assumption} & \\
& \leftrightarrow & \mathbf{N}_1 \models \mathscr{Q}x(\chi, U(x, y_1, \ldots, y_n))[s] & \\
& \leftrightarrow & \mathbf{M} \models \mathscr{Q}x(\varphi', \psi)[s] & \text{(DEXT).}
\end{array}$$

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