MIX

Sylvain Salvati

INRIA Bordeaux Sud-Ouest

MCFG+2

Outline

Outline

MIX

$$MIX = \{w \in \{a; b; c\}^* || w|_a = |w|_b = |w|_c\}$$

The Bach language

- Bach (1981)

Exercise 2: Let $L = \{X | X = (abc)^n\}$. L is CF (in fact regular).
But Scramble (L) is not CF. For let $L' = \{X | X = a^n b^m c^k\}$ then
$L' \cap L = \{X | X = a^n b^n c^m\}$ is not CF, but since the intersection of
a CF language and a regular language is CF, L can't be CF.

Wikipedia entry:
`http://en.wikipedia.org/wiki/Bach_language`

The *MIX* language

- Marsh (1985)

Conjecture: *MIX* is not an indexed language.

Proof. Consider the language MIX = SCRAMBLE($(abc)^+$) (the names 'mix' and 'MIX' — pronounced 'little mix' and 'big mix' were the happy invention of Bill Marsh; 'little mix' is the scramble of $(ab)^+$).

MIX and Tree Adjoining Grammars

- Joshi (1985)

[*MIX*] represents the extreme case of the degree of free word order permitted in a language. This extreme case is linguistically not relevant. [. . .] TAGs also cannot generate this language although for TAGs the proof is not in hand yet.

MIX and Tree Adjoining Grammars

- Vijay Shanker, Weir, Joshi (1991)

of strings of equal number of a's, b's, and c's in any order. MIX can be regarded as the extreme case of free word order. It is not known yet whether TAG, HG, CCG and LIG can generate MIX. This has turned out to be a very difficult problem. In fact, it is not even known whether an IG can generate MIX.

MIX and mildly context sensitive languages

- Joshi, Vijay Shanker, Weir (1991)

in MCSL; 2) languages in MCSL can be parsed in polynomial time; 3) MCSGs capture only certain kinds of dependencies, e.g., nested dependencies and certain limited kinds of crossing dependencies (e.g., in the subordinate clause constructions in Dutch or some variations of them, but perhaps not in the so-called MIX (or Bach) language, which consists of equal numbers of a's, b's, and c's in any order 4) languages in MCSL have constant growth property, i.e., if the strings of a language

Outline

Group languages

Group finite presentation:

- a finite set of generators Σ
- a finite set of defining equations E

Word problem: given w in Σ^*, is $w =_E 1$?
Group language: $\{w \in \Sigma^* \mid w =_E 1\}$

- the word problem is in general undecidable (Novikov 1955, Boone 1958)
- the languages of different representation of a group a rationally equivalent
- relate algebraic properties of groups to language-theoretic properties of their group languages

Example: a group language is context free iff its underlying group is virtually free (Muller Schupp 1983)

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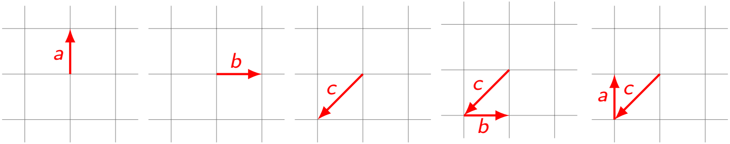
MIX as a group language

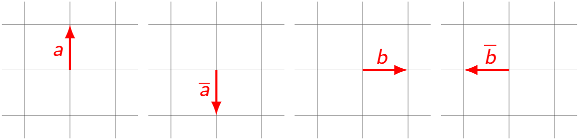
- Generators: $\{a; b; c\}$
- Defining equations: $a^{-1} = bc = cb$, $b^{-1} = ac = ca$, $c^{-1} = ab = ba$

$\mathbb{Z}^2$ is the group that has this presentation.



Yet another presentation of $\mathbb{Z}^2$

- Generators: $\{a; \overline{a}; b; \overline{b}\}$
- Defining equations: $a^{-1} = \overline{a}$, $b^{-1} = \overline{b}$, $ab = ba$, $a\overline{b} = \overline{b}a$, $\overline{a}b = b\overline{a}$, $\overline{a}\,\overline{b} = \overline{b}\,\overline{a}$



The associated group language is

$$O_2 = \{w \in \{a; \overline{a}; b; \overline{b}\}^* \,|\, |w|_a = |w|_{\overline{a}} \wedge |w|_b = |w|_{\overline{b}}\}$$

MIX and O_2: group languages of $\mathbb{Z}^2$

MIX and O_2 are rationally equivalent

MIX and computational group theory

- Gilman (2005)

is indexed but not context free seems to have been open for several years. It does not even seem to be known whether or not the word problem of $Z \times Z$ is indexed.

Outline

A 2-MCFG for O_2

$$
\begin{array}{c}
S(xy) \leftarrow Inv(x, y) \\
\hline
Inv(x_1 y_1, y_2 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
Inv(x_1 x_2 y_1, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
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Inv(y_1, y_2 x_1 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
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Inv(\alpha x_1 \overline{\alpha}, x_2) \leftarrow Inv(x_1, x_2) \\
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Inv(\alpha x_1, x_2 \overline{\alpha}) \leftarrow Inv(x_1, x_2) \\
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\hline
Inv(\epsilon, \epsilon) \leftarrow
\end{array}
$$

where $\alpha \in \{a; b\}$

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

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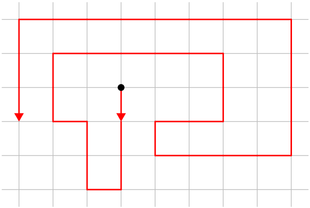
Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

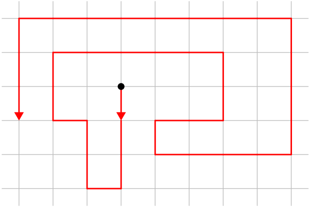
A graphical interpretation of O_2.

Graphical interpretation of the word $\bar{a}\bar{a}\bar{a}\bar{b}aa\bar{b}aabbbbb\bar{a}\bar{a}\bar{b}\bar{b}\bar{a}bbbbaaaa\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{a}\bar{a}\bar{a}$:

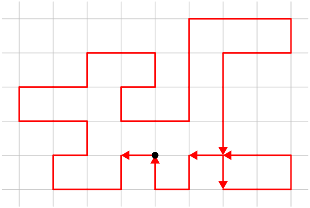


A graphical interpretation of O_2.

Graphical interpretation of the word $\overline{aaa}\bar{b}aa\bar{b}aabbbbb\overline{aa}\overline{bb}\bar{a}bbbbaaaa\overline{bbbbbbbb}\overline{aaa}$:

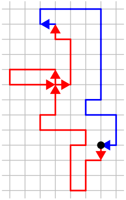


The words in O_2 are precisely the words that are represented as closed curves:
$\bar{b}\bar{a}\overline{bb}aba\overline{bb}abbabb\bar{a}\bar{b}\bar{a}bbaaabbb\bar{a}\overline{bb}\overline{aaaa}bba\overline{bbb}\bar{a}\bar{b}a$

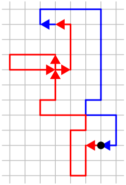


Parsing with the grammar

Rule $Inv(\bar{a}x_1a, x_2) \leftarrow Inv(x_1, x_2)$



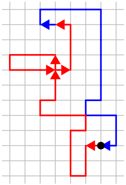
$Inv(\bar{a}\bar{b}\bar{a}\bar{a}\bar{b}aaaba\bar{b}\bar{b}\bar{b}abaaa\bar{b}\bar{b}\bar{b}\bar{a}bbbbaaa\bar{b}a, \bar{b}abbbb\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{a}bb\bar{a}\bar{a}\bar{b})$



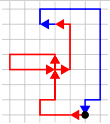
$Inv(\bar{b}\bar{a}\bar{a}\bar{b}aaaba\bar{b}\bar{b}\bar{b}abaaa\bar{b}\bar{b}\bar{b}\bar{a}bbbbaaa\bar{b}, \bar{b}abbbb\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{a}bb\bar{a}\bar{a}\bar{b})$

Parsing with the grammar

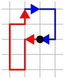
Rule: $Inv(x_1 y_1, y_2 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$



$Inv(\overline{b}\,\overline{aa}\,\overline{b}aaaba\overline{bbb}abaaa\overline{bbb}\,\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\,\overline{b}\,\overline{a}bb\overline{aa}\,\overline{b})$



$Inv(\overline{bbb}abaaa\overline{bbb}\,\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\,\overline{b}\,\overline{a})$



$Inv(\overline{b}\,\overline{aa}\,\overline{b}aaaba, bb\overline{aa}\,\overline{b})$

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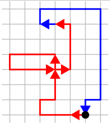
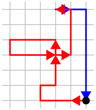


$Inv(\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\overline{b}\overline{a})$



$Inv(\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, bb\overline{aaaaaa})$

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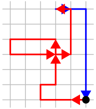

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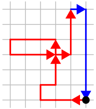

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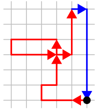
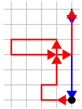


$Inv(\overline{b}\overline{b}\overline{b}abaaa\overline{b}\overline{b}\overline{b}\overline{a}bbbbaaa, b\overline{a}\overline{a}\overline{a}\overline{a}\overline{a}\overline{a})$



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Parsing with the grammar

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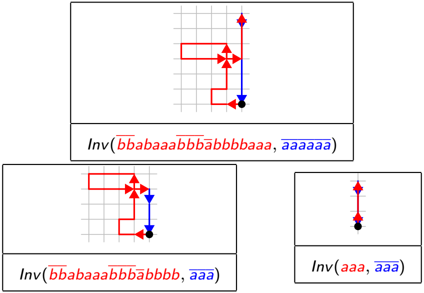

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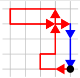

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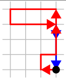

$Inv(\bar{b}abaaa\bar{b}\bar{b}\bar{b}\bar{a}bbb, \bar{a}\bar{a}\bar{a})$

Outline

The proof of the Theorem

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

The proof is done by induction on the lexicographically ordered pairs
$(|w_1 w_2|, \max(|w_1|, |w_2|))$.
There are five cases:

Case 1: w_1 or w_2 equal ϵ:

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Case 1: w_1 or w_2 equal ϵ:
w.l.o.g., $w_1 \neq \epsilon$, then by induction hypothesis, for any v_1 and v_2 different from ϵ such that $w_1 = v_1 v_2$, $Inv(v_1, v_2)$ is derivable then:

$$\frac{Inv(v_1, v_2) \quad Inv(\epsilon, \epsilon)}{Inv(v_1 v_2 = w_1, \epsilon)} \; Inv(x_1 x_2 y_1, y_2) \leftarrow Inv(x_1, x_2), \; Inv(y_1, y_2)$$

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Case 2: $w_1 = s_1 w_1' s_2$ and $w_2 = s_3 w_2' s_4$ and for $i, j \in \{1; 2; 3; 4\}$, s.t. $i \neq j$, $\{s_i; s_j\} \in \{\{a; \overline{a}\}; \{b; \overline{b}\}\}$:

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e.g., if $i = 1$, $j = 2$, $s_1 = a$ and $s_2 = \overline{a}$ then by induction hypothesis $Inv(w_1', w_2)$ is derivable and:

$$\frac{Inv(w_1', w_2)}{Inv(a w_1' \overline{a}, w_2)} \; Inv(a x_1 \overline{a}, x_2) \leftarrow Inv(x_1, x_2)$$

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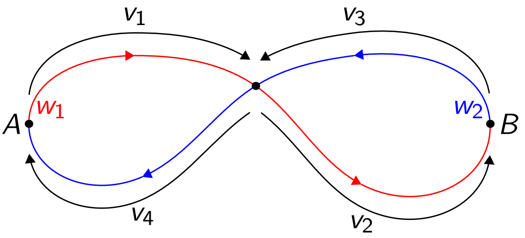

$$\frac{Inv(v_1, v_4) \quad Inv(v_2, v_3)}{Inv(v_1 v_2 = w_1, v_3 v_4 = w_2)}$$

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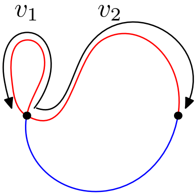

$$\frac{Inv(v_1, \epsilon) \quad Inv(v_2, w_2)}{Inv(v_1 v_2 = w_1, w_2)}$$

The proof of the Theorem

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There are five cases:

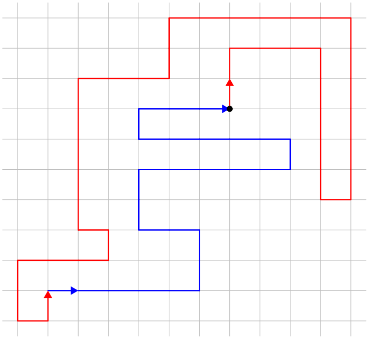
Case 5: w_1 and w_2 do not start or end with compatible letters, the curve representing then do not intersect and do not start or end with a loop.

Case 5

No rule other than

$$\begin{aligned} Inv(x_1 y_1 x_2, y_2) &\leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\ Inv(x_1, y_1 x_2 y_2) &\leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \end{aligned}$$

can be used.



Case 5

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$$
\begin{array}{rcl}
Inv(x_1 y_1 x_2, y_2) & \leftarrow & Inv(x_1, x_2), Inv(y_1, y_2) \\
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\end{array}
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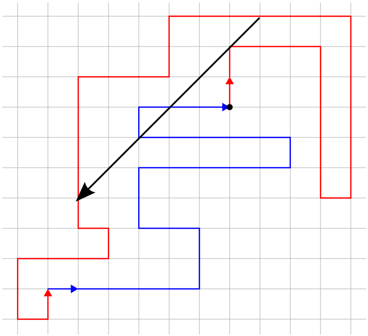
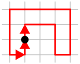
can be used.



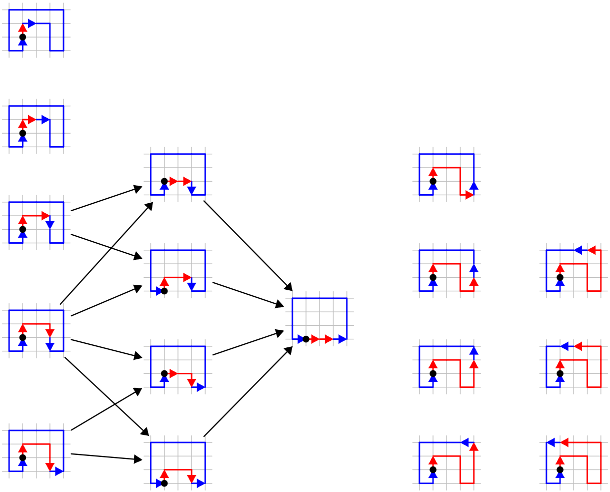
The relevance of case 5

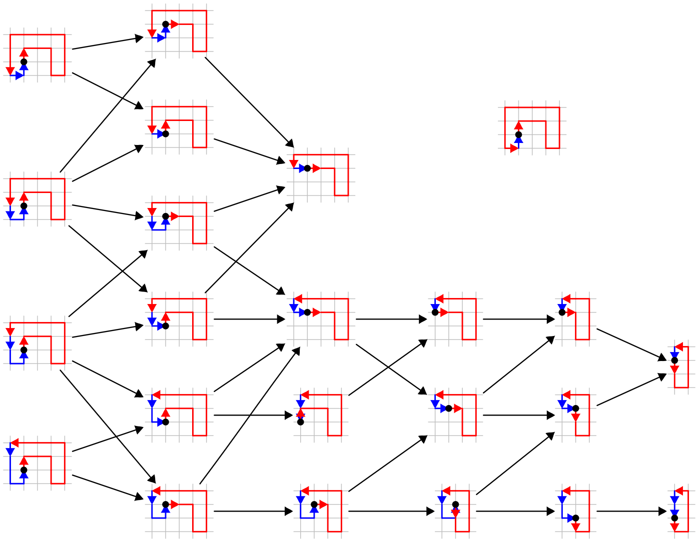
The word

$$abb\bar{a}\bar{a}baaa\bar{b}\bar{b}\bar{b}\bar{b}\bar{a}\bar{a}\bar{a}ba$$

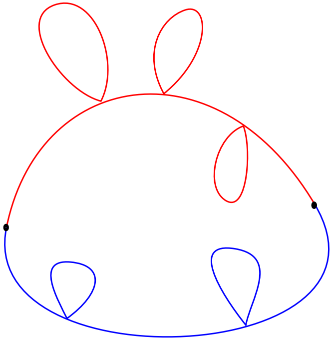


is not in the language of the grammar only containing the well-nested rules.





Solving case 5: towards geometry



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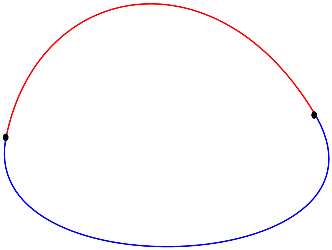

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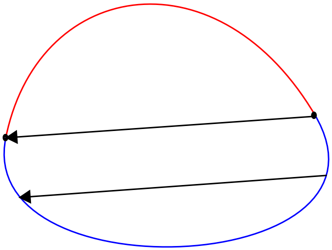

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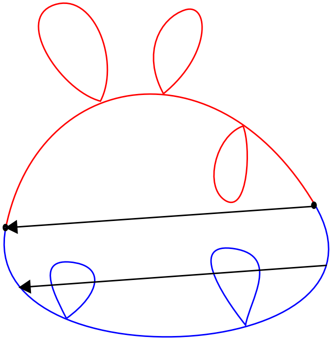
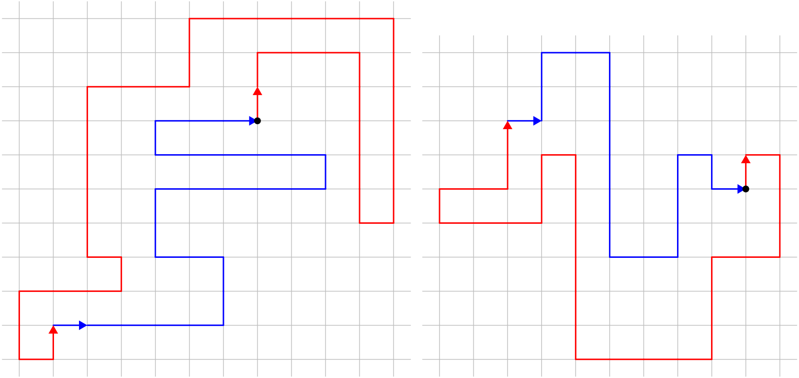


Solving case 5: a geometrical invariant



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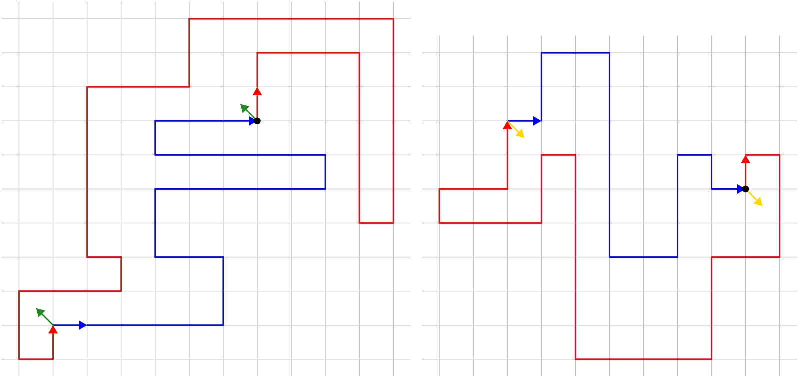
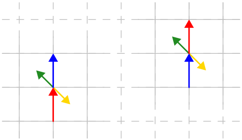


Solving case 5: a geometrical invariant

An invariant on the Jordan curve representing $w_1' w_2'$:



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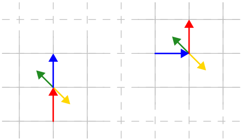

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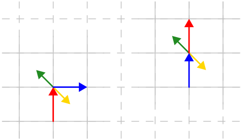

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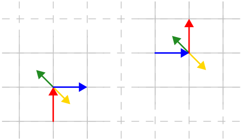

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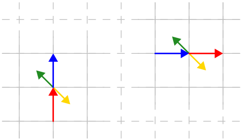

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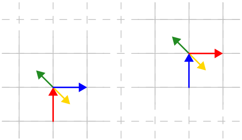
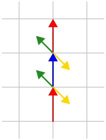


$w_1' = aw_1''b$ and $w_2' = bw_2''a$



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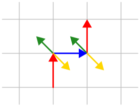

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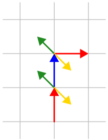	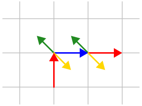
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Outline

On Jordan curves

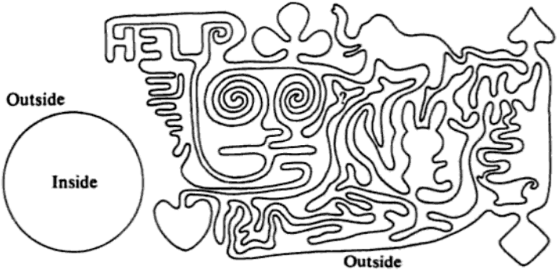

Figure 13.1 Two Jordan curves.

illustration from: A combinatorial introduction to topology by Michael Henle (Dover Publications).

Theorem: There is $k \in \{-1; 1\}$ such that the winding number of Jordan curve around a point in its interior is k, its winding number around a point in its exterior is 0.

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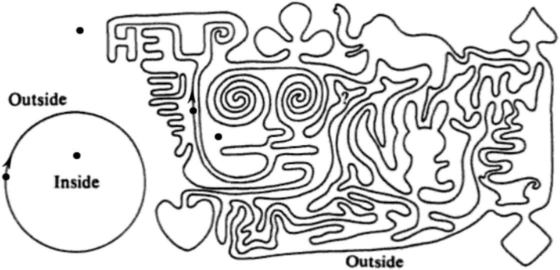

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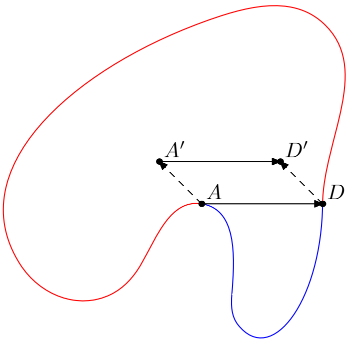
A theorem on Jordan curves

Theorem: If A and D are two points on a Jordan curve J such that there are two points A' and D' inside J such that $\overrightarrow{AD} = \overrightarrow{A'D'}$, then there are two points B and C pairwise distinct from A and D such that A, B, C, and D appear in that order on one of the arcs going from A to D and $\overrightarrow{AD} = \overrightarrow{BC}$.



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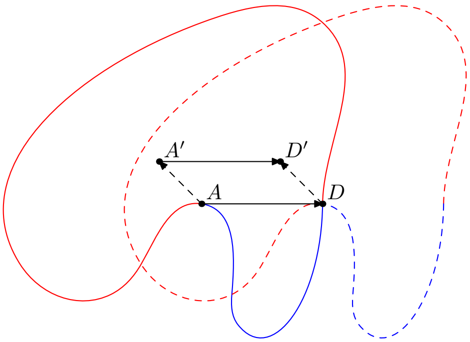

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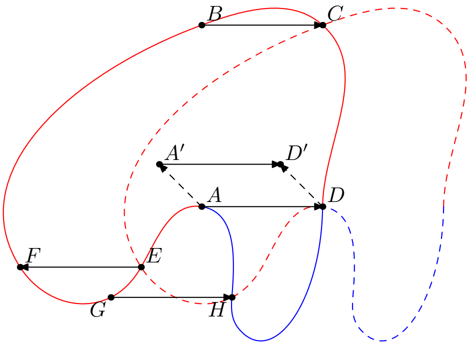

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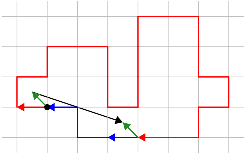
Applying this Theorem solves case 5.



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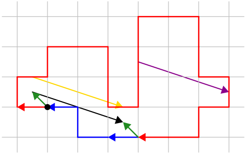

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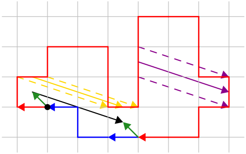
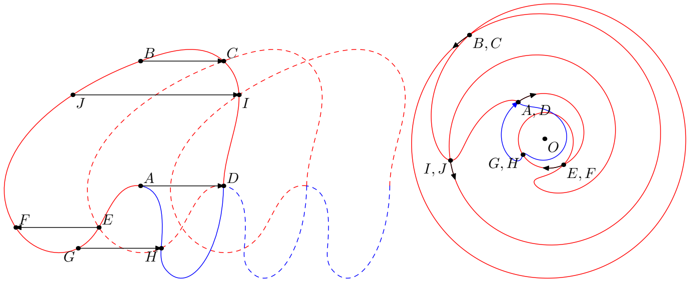


Simple curves, translations, intersections and the complex exponential

Let's suppose that $D - A = 1$

let $\varphi : \begin{cases} \mathbb{C} & \to & \mathbb{C} - \{0\} \\ z & \to & e^{2i\pi z} \end{cases}$.



φ transforms arcs performing translation of k into arc that have k as winding number around 0.

Simple curves, translations, intersections and the complex exponential

Let's suppose that $D - A = 1$ and that $A_0 = A' = 0$, $A_1 = D' = 1, \dots, A_k = k$

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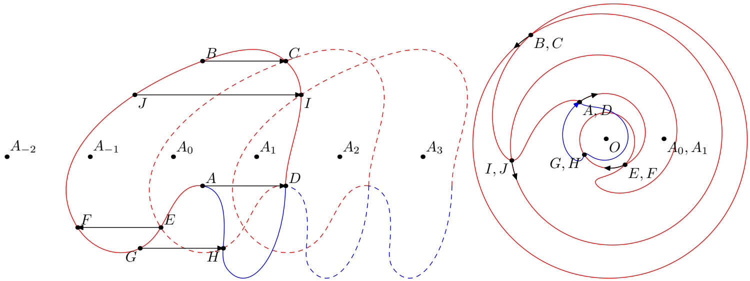

φ sums up the winding number of a Jordan curve around the A_i's as the winding number around $\varphi(A_0) = \varphi(0) = 1$.

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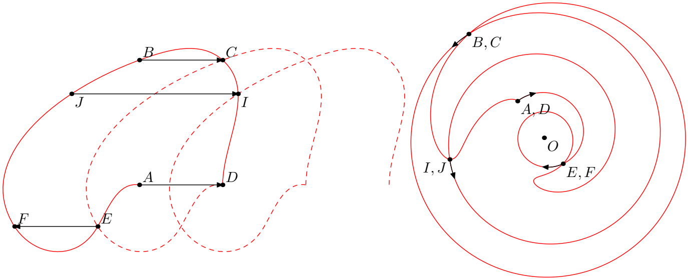

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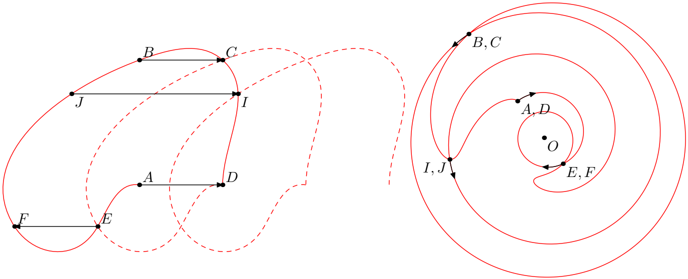

Main Lemma: a simple arc J from A to D (*resp.* D to A) does not contain B and C as required in the Theorem iff $\varphi(J)$ is a Jordan curve of $\mathbb{C} - \{0\}$ that belong to the homotopy class 1 (*resp.* -1).

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Proof.

Let $wn(J, z)$ be the winding number of a closed curve around z.

- If $\varphi(J)$ contains a closed subarc J' with $wn(J, 0) = k$, then it corresponds to a subarc of J going from a point E to $E + k$,
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Corollary: a simple path J from A to D (*resp.* D to A) does not contain B and C as required in the Theorem iff $\varphi(J)$ is a Jordan curve of $\mathbb{C} - \{1\}$ that belong to the homotopy class 0 or 1 (*resp.* or -1).

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Proving the Theorem

Corollary: if J is a simple closed curve of $\mathbb{C}$ composed with two curves J_1 and J_2 respectively going from A to D and D to A which do not contain points B and C as required in the Theorem then $|wn(\varphi(J),1)| = |wn(\varphi(J_1),1) + wn(\varphi(J_2),1)| \leq 1$.

Lemma: if J is a simple closed curve of $\mathbb{C}$ composed with two curves J_1 and J_2 respectively going from A to D and D to A such that 0 and 1 are in the interior of J, then either $|wn(\varphi(J),1)| \geq 2$.

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Outline

Conclusion

- we have showed that O_2 is a 2-MCFL exhibiting the first non-virtually free group language that is proved to belong to an interesting class of language,
- this implies that contrary to the usual conjecture we have showed that *MIX* is a 2-MCFLs.

Conjectures

Well-nestedness:

$$\frac{\begin{array}{c}\text{Well-nested}\\ Inv(y_1x_1x_2, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)\end{array}}{\begin{array}{c}\text{Not well-nested}\\ Inv(y_1x_1y_2, x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)\end{array}}$$

MCFG_{wn} are MCFGs with well-nested rules.

- MCFL_{wn} coincide with non-duplicating IO/OI,
- MCFL is incomparable with IO or OI.

Thus the following conjectures:

- mildly context sensitive languages may well be, as advocated by Kanazawa, MCFL_{wn}
- O_2 and MIX should not be a MCFL_{wn}
- semilinear rational cones included in OI should be included in MCFL_{wn}
- O_2 and MIX should not be in OI.

Open question:

- Is O_3 an MCFL?

Conjectures

Well-nested:

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- Is O_3 an MCFL?

Conjectures

Well-nestedness:

Well-nested

$$\frac{Inv(y_1 x_1 x_2, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)}{\text{Not well-nested}}$$

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