**Ehrenfeucht-Fraïssé Theorem**

1. Definition. Let $K_n = \{a_1, \ldots, a_n\}$ be the first $n$ parameters. Let $M$ and $N$ be interpretations in universes $U$ and $V$, respectively, and let $c_1, \ldots, c_n \in U$ and $d_1, \ldots, d_n \in V$. Define $(M, c_1, \ldots, c_n) \simeq_k (N, d_1, \ldots, d_n)$ recursively by

   - $(M, c_1, \ldots, c_n) \simeq_0 (N, d_1, \ldots, d_n)$ iff for every atomic $K_n$-sentence $A$, $M \models A[c_1, \ldots, c_n] \Leftrightarrow N \models A[d_1, \ldots, d_n]$.
   - $(M, c_1, \ldots, c_n) \simeq_{k+1} (N, d_1, \ldots, d_n)$ iff the following two conditions hold:
     - for all $c \in U$, there exists a $d \in V$ such that $(M, c_1, \ldots, c_n, c) \simeq_k (N, d_1, \ldots, d_n, d)$, and
     - for all $d \in V$, there exists a $c \in U$ such that $(M, c_1, \ldots, c_n, c) \simeq_k (N, d_1, \ldots, d_n, d)$.

2. Definition. Define the *quantifier rank* of a formula $A$ by

$$\begin{aligned} \text{qr}(A) &= 0 && \text{if } A \text{ is atomic,} \\ \text{qr}(\neg A) &= \text{qr}(A), \\ \text{qr}(A\ b\ B) &= \max(\text{qr}(A), \text{qr}(B)) && (b \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}), \\ \text{qr}(qx\ A) &= \text{qr}(A) + 1 && (q \in \{\forall, \exists\}). \end{aligned}$$

3. Definition. The relation $(M, c_1, \ldots, c_n) \equiv_k (N, d_1, \ldots, d_n)$ holds iff for every $K_n$-sentence $A$ such that $\text{qr}(A) \leq k$, $M \models A[c_1, \ldots, c_n] \Leftrightarrow N \models A[d_1, \ldots, d_n]$.
4. Theorem. $(M, c_1, \ldots, c_n) \simeq_k (N, d_1, \ldots, d_n)$ iff $(M, c_1, \ldots, c_n) \equiv_k (N, d_1, \ldots, d_n)$.
5. Theorem. There is no sentence $B$ of first-order logic such that for all *finite* graphs $G$, $M_G \models B$ iff $G$ is connected.

   Compactness is useless to show first-order definability over finite structures. We use the Ehrenfeucht-Fraïssé theorem.

   Proof. Suppose such a sentence $B$ exists. Let $k = \text{qr}(B)$. To derive a contradiction, it suffices to give two finite graphs $G$ and $H$ such that

   - $G$ is connected,
   - $H$ is not connected, and
   - $M_G \simeq_k M_H$.

   Let $G = (V, E)$, where $V = \{v_1, \ldots, v_l\}$ and $E = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq l-1\} \cup \{\{v_l, v_1\}\}$. Let $H = (V \cup V', E \cup E')$, where $(V', E')$ is an isomorphic copy of $(V, E)$. Define

$$\text{dist}(v_i, v_j) = \begin{cases} j - i & \text{if } -l/2 \leq j - i \leq l/2, \\ j - i - l & \text{if } l/2 < j - i, \\ j - i + l & \text{if } j - i < -l/2, \end{cases}$$

   and likewise $\text{dist}(v'_i, v'_j)$ for $v'_i, v'_j \in V'$. If $v \in V$ and $v' \in V'$, we define $\text{dist}(v, v') = \infty$ and $\text{dist}(v', v) = -\infty$.

   Suppose $l \geq 2^k$. We can prove

$$(M_G, c_1, \ldots, c_n) \simeq_{k-n} (M_H, d_1, \ldots, d_n)$$

   for all $n \in \{0, \ldots, k\}$ if the following conditions hold of all $i, j \in \{1, \ldots, n\}$:

   - $|\text{dist}(c_i, c_j)| \leq 2^{k-n}$ implies $\text{dist}(c_i, c_j) = \text{dist}(d_i, d_j)$, and
   - $|\text{dist}(d_i, d_j)| \leq 2^{k-n}$ implies $\text{dist}(c_i, c_j) = \text{dist}(d_i, d_j)$.