First-Order Logic

1. Language of first-order logic
   - individual symbols
     - individual variables: $x_1, x_2, x_3, \ldots, x, y, z, \ldots$
     - individual parameters: $a_1, a_2, a_3, \ldots, a, b, c, \ldots$
   - n-ary predicates ($n \geq 1$)
   - inductive definition of *formulas*:

   (a) If P is an n-ary predicate and $c_1, \ldots, c_n$ are individual symbols, then

   $$Pc_1 \ldots c_n$$

   is a formula.

   (b) If A is a formula, then $\neg A$ is a formula.

   (c) If A and B are formulas, then

   $$(A \wedge B) \quad (A \vee B) \quad (A \rightarrow B) \quad (A \leftrightarrow B)$$

   are formulas.

   (d) If A is a formula and x is an individual variable, then

   $$\forall x\, A \quad \exists x\, A$$

   are formulas.

   A formula is *pure* if it contains no parameters.

2. Examples. Suppose P and R are unary (i.e., 1-ary) predicates, and Q is a binary (i.e., 2-ary) predicate. The following are pure formulas:

$$\forall x\, Px \vee \neg \exists y\, Qxy$$
$$\forall x\, Px \rightarrow (\forall x\, Qxy \vee Rx)$$
$$\forall x (Px \rightarrow (Qxy \vee Rx))$$
$$\forall x (Px \rightarrow \forall x (Qxy \wedge Rx))$$

3. Free variables:

$$\begin{aligned} \mathrm{FV}(Pc_1 \ldots c_n) &= \{\, c_i \mid 1 \leq i \leq n, c_i \text{ is a variable} \,\} \\ \mathrm{FV}(\neg A) &= \mathrm{FV}(A), \\ \mathrm{FV}(A\, b\, B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \quad (b \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}) \\ \mathrm{FV}(qx\, A) &= \mathrm{FV}(A) - \{x\} \quad (q \in \{\forall, \exists\}) \end{aligned}$$

   A formula A is *closed* if $\mathrm{FV}(A) = \varnothing$. A closed formula is also called a *sentence*. The set of all closed pure formulas is written $\mathbb{F}$.

4. Substitution of a parameter a for a variable x:

$$\begin{aligned} x[a/x] &= a, \\ c[a/x] &= c \quad \text{if } c \text{ is not } x, \\ (Pc_1 \ldots c_n)[a/x] &= Pc_1[a/x] \ldots c_n[a/x], \\ (\neg A)[a/x] &= \neg A[a/x], \\ (A\, b\, B)[a/x] &= A[a/x]\; b\; B[a/x], \\ (qx\, A)[a/x] &= qx\, A, \\ (qy\, A)[a/x] &= qy\, A[a/x] \quad \text{if } y \text{ is not } x. \end{aligned}$$

   Example:

$$(\forall x\, Px \vee \neg \exists y\, Qxy)[a/x] = \forall x\, Px \vee \neg \exists y\, Qay.$$

5. Formulas with constants in U. Let U be a non-empty set (*universe* or *domain*). A U-formula is like a formula except that instead of parameters, elements of U (called *individual constants*) are allowed as individual symbols. A U-formula may not contain any parameters. A closed U-formula is called a *U-sentence*. Let $\mathbb{F}^U$ be the set of all U-sentences.
6. Substitution of an individual constant k for a variable x: $A[k/x]$.
7. An *interpretation* of $\mathbb{F}$ in a universe U is a function M which assigns each n-ary predicate P an n-ary relation P^M on U.
8. For each U-sentence A, we define what it means for A to be *true* under an interpretation M in U. We write $M \models A$ for "A is true under M".
   - $M \models Pe_1 \dots e_n$ iff $(e_1, \dots, e_n)$ stands in the relation P^M.
   - $M \models \neg A$ iff $M \not\models A$.
   - $M \models A \wedge B$ iff $M \models A$ and $M \models B$.
   - $M \models A \vee B$ iff at least one of $M \models A$ and $M \models B$ holds.
   - $M \models A \rightarrow B$ iff at least one of $M \neg \models A$ and $M \models B$ holds.
   - $M \models A \leftrightarrow B$ iff either $M \models A$ and $M \models B$ or $M \not\models A$ and $M \not\models B$.
   - $M \models \forall x\, A$ iff for all $d \in U$, $M \models A[d/x]$.
   - $M \models \exists x\, A$ iff for at least one $d \in U$, $M \models A[d/x]$.
9. Let M be an interpretation, A a (signed) pure sentence, and S be a set of (signed) pure sentences.
   - M *satisfies* A iff A is true under M.
   - A is *satisfiable in* U iff at least one interpretation in U satisfies A.
   - A is *satisfiable* iff A is satisfiable in some universe.
   - M *satisfies* S iff M satisfies all A in S.
   - M is a *model of* S iff M satisfies S.
   - S is *satisfiable in* U iff some interpretation in U satisfies S.
   - S is *satisfiable* iff S is satisfiable in some universe.
   - A is *valid* in U iff A is true under every interpretation in U.
   - A is *valid* iff A is valid in every universe.
   - A is a *logical consequence* of S iff every interpretation (in any universe) that satisfies S satisfies A.
   - A is *logically equivalent to* B iff A and B are true under the same interpretations (in any universe).
10. Sentences with parameters. Let $A(a_1, \dots, a_n)$ be a (signed) sentence containing exactly $a_1, \dots, a_n$ as parameters. Let M be an interpretation in U.
    - $A(a_1, \dots, a_n)$ is *satisfiable under* M iff there is an n-tuple $(k_1, \dots, k_n)$ of elements of U such that $A(k_1, \dots, k_n)$ is true under M.
    - $A(a_1, \dots, a_n)$ is *valid under* M iff for every n-tuple $(k_1, \dots, k_n)$ of elements of U, $A(k_1, \dots, k_n)$ is true under M.
    - $A(a_1, \dots, a_n)$ is *satisfiable in* U iff it is satisfiable under at least one interpretation in U.
    - $A(a_1, \dots, a_n)$ is *valid in* U iff it is valid under every interpretation in U.
    - $A(a_1, \dots, a_n)$ is *satisfiable* iff it is satisfiable in at least one universe.
    - $A(a_1, \dots, a_n)$ is *valid* iff it is valid in every universe.

    (Here, $A(k_1, \dots, k_n)$ is just like $A(a_1, \dots, a_n)$ except that k_i occurs in place of a_i for $i = 1, \dots, n$.)
11. Let S be a set of (signed) sentences with parameters. If σ is a function from the set of parameters of S (i.e., parameters that occur in S) to U, then $A\sigma$ is the result of replacing each parameter a by $\sigma(a)$ in A.

- S is *(simultaneously) satisfiable in U* if there exists an interpretation M in U and a function σ from the set of parameters of S to U such that for every $A \in S$, $A\sigma$ is true under M.
- S is *(simultaneously) satisfiable* if S is (simultaneously) satisfiable in some universe.

Tableaux for First-Order Logic

1. Tableau expansion rules:

$$\frac{T\,\forall x\,A}{T\,A[a/x]} \qquad \frac{F\,\forall x\,A}{F\,A[b/x]} \qquad\qquad \frac{T\,\exists x\,A}{T\,A[b/x]} \qquad \frac{F\,\exists x\,A}{F\,A[a/x]}$$

   Proviso: b is a new parameter.

2. Example: $\forall x(Px \to Qx) \to (\forall x\,Px \to \forall\,xQx)$.
3. Example: $\exists y(\exists x\,Px \to Py)$.
4. Exercise. Find tableau proofs of the following formulas:

$$\begin{aligned}
&\forall y(\forall x\,Px \to Py)\\
&\forall x\,Px \to \exists x\,Px\\
&\exists y(Py \to \forall x\,Px)\\
&\neg\exists y\,Py \to (\forall y(\exists x\,Px \to Py))\\
&\exists x\,Px \to \exists x\,Py\\
&\forall x(Px \wedge Qx) \leftrightarrow \forall x\,Px \wedge \forall x\,Qx\\
&(\forall x\,Px \vee \forall x\,Qx) \to \forall x(Px \vee Qx)\\
&\exists(Px \vee Qx) \leftrightarrow (\exists x\,Px \vee \exists x\,Qx)\\
&\exists x(Px \wedge Qx) \to (\exists x\,Px \wedge \exists x\,Qx)\\
&\exists x(Px \to Qx) \leftrightarrow (\forall x\,Px \to \exists x\,Qx)
\end{aligned}$$

5. Exercise. Let C be a closed formula. Find tableau proofs of the following formulas:

$$\begin{aligned}
&\forall x(Px \vee C) \leftrightarrow (\forall x\,Px \vee C)\\
&\exists x(Px \wedge C) \leftrightarrow (\exists x\,Px \wedge C)
\end{aligned}$$

6. Exercise. Find a tableau proof of $(H \wedge K) \to L$, where

$$\begin{aligned}
H &= \forall x\forall y(Rxy \to Ryx)\\
K &= \forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)\\
L &= \forall x\forall y(Rxy \to Rxx)
\end{aligned}$$

7. Exercise. Find a tableau proof of $(A \wedge B) \to C$, where

$$\begin{aligned}
A &= \forall x((Fx \wedge Gx) \to Hx) \to \exists x(Fx \wedge \neg Gx)\\
B &= \forall x(Fx \to Gx) \vee \forall x(Fx \to Hx)\\
C &= \forall x((Fx \wedge Hx) \to Gx) \to \exists x(Fx \wedge Gx \wedge \neg Hx)
\end{aligned}$$

8. Lemma. Let S be a set of signed sentences with parameters.
   (a) If S is satisfiable and $T\,\forall x\,A \in S$, then for every parameter a, $S \cup \{T\,A[a/x]\}$ is satisfiable.
   (b) If S is satisfiable and $F\,\exists x\,A \in S$, then for every parameter a, $S \cup \{F\,A[a/x]\}$ is satisfiable.
   (c) If S is satisfiable, $F\,\forall x\,A \in S$, and b is a parameter that does not occur in any element of S, then $S \cup \{F\,A[b/x]\}$ is satisfiable.
   (d) If S is satisfiable, $T\,\exists x\,A \in S$, and b is a parameter that does not occur in any element of S, then $S \cup \{T\,A[b/x]\}$ is satisfiable.

9. Soundness Theorem. Every satisfiable set of signed formulas is consistent (i.e., has no finite closed tableau).

10. A set S of signed U-sentences is said to *obey*

    - $\dfrac{T\,\forall x\,A}{T\,A[a/x]}$ relative to U if whenever $T\,\forall x\,A \in S$, then $T\,A[k/x] \in S$ for all $k \in U$.
    - $\dfrac{F\,\forall x\,A}{F\,A[b/x]}$ relative to U if whenever $F\,\forall x\,A \in S$, then $F\,A[k/x] \in S$ for at least one element k of U.
    - $\dfrac{T\,\exists x\,A}{T\,A[b/x]}$ relative to U if whenever $T\,\exists x\,A \in S$, then $T\,A[k/x] \in S$ for at least one element k of U.
    - $\dfrac{F\,\exists x\,A}{F\,A[a/x]}$ relative to U if whenever $F\,\exists x\,A \in S$, then $F\,A[k/x] \in S$ for all $k \in U$.

11. A set S of signed U-sentences is a *Hintikka set* for a universe U iff

    - There are no n-ary predicate and elements $k_1, \ldots, k_n$ of U such that both $T\,Pk_1 \ldots k_n$ and $F\,Pk_1 \ldots k_n$ are in S, and
    - S obeys all tableau expansion rules (relative to U).

12. Lemma. Every Hintikka set for a universe U is satisfiable in U.

    Proof. Define an interpretation M in U by

    $$P^M \text{ holds of } (k_1, \ldots, k_n) \text{ iff } T\,Pk_1 \ldots k_n \in S.$$

    We show by induction that for all U-sentences A, $T\,A \in S$ implies $M \models A$ and $F\,A \in S$ implies $M \not\models A$.

    Induction Basis. A is $Pk_1 \ldots k_n$ for some n-ary predicate P and $k_1, \ldots, k_n \in U$. If $T\,Pk_1 \ldots k_n \in S$, then $M \models Pk_1 \ldots k_n$ by the definition of M. If $F\,Pk_1 \ldots k_n \in S$, then $T\,Pk_1 \ldots k_n \notin S$, since S is a Hintikka set. So $M \not\models Pk_1 \ldots k_n$ by the definition of M.

    Induction Step. Case 6. A is $\forall x\,B$. If $T\,\forall x\,B \in S$, then for all $k \in U$, $T\,B[k/x] \in S$ since S is a Hintikka set. By induction hypothesis, $M \models B[k/x]$. Since this holds of all k, $M \models \forall x\,B$. If $F\,\forall x\,B \in S$, then there exists a $k \in U$ such that $F\,B[k/x] \in S$, since S is a Hintikka set. By induction hypothesis, $M \not\models B[k/x]$. Therefore, $M \not\models \forall x\,B$.

    Case 7. Similar.

13. Let $a_1, a_2, a_3, \ldots$ list the parameters. We say that a node labeled X in a tableau is *fulfilled* for n iff for every open path ρ that goes through the node, the following hold:

    - if $\dfrac{X}{Y}$ is an instance of a tableau expansion rule in propositional logic, Y is on ρ,
    - if $\dfrac{X}{\begin{array}{c}Y_1\\Y_2\end{array}}$ is an instance of a tableau expansion rule, both Y_1 and Y_2 are on ρ,
    - if $\dfrac{X}{Y \mid Z}$ is an instance of a tableau expansion rule, either Y or Z is on ρ,
    - if $\dfrac{X}{\begin{array}{c|c}Y_1 & Z_1\\Y_2 & Z_2\end{array}}$ is an instance of a tableau expansion rule, either both Y_1 and Y_2 are on ρ, or else both Z_1 and Z_2 are on ρ,
    - if $X = T\,\forall x\,A$, then $T\,A[a_i/x]$ is on ρ for all $i \leq n$,
    - if $X = F\,\exists x\,A$, then $F\,A[a_i/x]$ is on ρ for all $i \leq n$,
    - if $X = F\,\forall x\,A$, then $F\,A[b/x]$ is on ρ for some b, and
    - if $X = T\,\exists x\,A$, then $T\,A[b/x]$ is on ρ for some b.

14. Lemma. Any finite tableau $\mathcal{T}$ for S can be extended to a finite tableau $\mathcal{T}'$ for S in which every node of $\mathcal{T}$ is fulfilled for n.

15. A tableau is *finished* if every open path obeys all tableau expansion rules relative to the set of parameters.

16. Lemma. For every set S of signed sentences, there is a finished tableau for S.

    Proof. Let $X_0, X_1, X_2, \ldots$ be an enumeration of the elements of S. We recursively define finite tableaux $\mathcal{T}_n$ for S, for all natural numbers n. Let $\mathcal{T}_0$ be a tableau with just one node, labeled by X_0. Now assume that $\mathcal{T}_n$ has been defined. We take all the nodes of $\mathcal{T}_n$ that are not yet fulfilled for n, and extend $\mathcal{T}_n$ to $\mathcal{T}'_n$ by fulfilling those nodes for n. Then we adjoin a new node labeled by X_{n+1} at the end of every open path in $\mathcal{T}'_n$. The result is $\mathcal{T}_{n+1}$. Having defined an infinite sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of finite tableaux for S, we let $\mathcal{T} = \bigcup_n \mathcal{T}_n$. It is easy to see that $\mathcal{T}$ is a tableau, every node of $\mathcal{T}$ is fulfilled for all n, and X_n is on every open path of $\mathcal{T}$ for all n. Therefore, $\mathcal{T}$ is a finished tableau for S.

17. Completeness Theorem. Every consistent set of signed sentences in first-order logic is satisfiable.

18. Compactness Theorem. Let S be a set of signed sentences. If every finite subset of S is satisfiable, then S is satisfiable.

19. Example of an application of the Compactness Theorem. Let R be a binary predicate. Every (finite or infinite) graph $G = (V, E)$ determines an interpretation M_G in universe V such that R^{M_G} holds of (v_1, v_2) iff $\{v_1, v_2\} \in E$. Show that there is no sentence A of first-order logic such that $M_G \models A$ iff G is connected.

    Consider $S' = \{A, \exists x((Rax \land \neg Rbx) \lor (Rxa \land \neg Rxb))\} \cup S$, where

    $$\begin{aligned} S &= \{\, D_n \mid n \geq 0 \,\}, \\ D_0 &= \neg Rab, \\ D_n &= \neg \exists x_1 \ldots \exists x_n (Rax_1 \land Rx_1x_2 \land \ldots Rx_{n-1}x_n \land Rx_nb) \quad \text{for } n \geq 1. \end{aligned}$$