Kőnig's Lemma and the Compactness Theorem for Propositional Logic

1. A *tree domain* is a subset D of the set of finite sequences of positive integers satisfying the following condition:

   - if $\langle i_1, \dots, i_n, k \rangle \in D$, then $\langle i_1, \dots, i_n \rangle \in D$, and for all $j < k$, $\langle i_1, \dots, i_n, j \rangle \in D$.

   If $v = \langle i_1, \dots, i_n \rangle$ and $w = \langle j_1, \dots, j_m \rangle$, then we write $v \frown w$ for $\langle i_1, \dots, i_n, j_1, \dots, j_m \rangle$.

2. A *labeled ordered tree* is $\langle D, L, \ell \rangle$, where

   - D is a tree domain (the set of *nodes*),
   - L is a set of *labels*, and
   - $\ell \colon D \to L$.

   In a labeled ordered tree $\langle i_1, \dots, i_n, k \rangle$ is the k-th *child* of $\langle i_1, \dots, i_n \rangle$. $\langle\rangle$ (the empty sequence) is the *root*. The *level* of a node $\langle i_1, \dots, i_n \rangle$ is n. (Thus the level of the root is 0.) A *path* is a (finite or infinite) sequence of nodes of the form

   $$\langle\rangle, \langle i_1 \rangle, \langle i_1, i_2 \rangle, \dots.$$

   In this course, a path is assumed to be *maximal*—that is to say, when a path is finite, it is assumed that the last node is a *leaf* (i.e., a node without children). A tree is *finitely branching* iff all nodes in it have only finitely many children.

3. Let $\mathcal{T} = \langle D, L, \ell \rangle$ be a labeled ordered tree, and let $w = \langle i_1, \dots, i_n \rangle$ be a node of $\mathcal{T}$. The *subtree of $\mathcal{T}$ rooted at w* is $\mathcal{T}' = \langle D', L, \ell' \rangle$, where

   $$D' = \{ v \mid w \frown v \in D \},$$
   $$\ell'(v) = \ell(w \frown v).$$

4. Kőnig's Lemma. Every infinite tree that is finitely branching has an infinite path.

   Proof. Let $\mathcal{T}$ be a finitely branching, infinite tree. For each natural number n, we define a node v_n of $\mathcal{T}$ such that the subtree of $\mathcal{T}$ rooted at v_n is infinite. The sequence $v_0, v_1, v_2, \dots$ will be an infinite path of $\mathcal{T}$. First we let $v_0 = \langle\rangle$ (the root of $\mathcal{T}$). The subtree of $\mathcal{T}$ rooted at v_0 is $\mathcal{T}$ itself, and so is infinite by assumption. Assume that v_n has been defined, and the subtree of $\mathcal{T}$ rooted at v_n is infinite. Since v_n has only finitely many children, there must be a child node w of v_n such that the subtree rooted at w is infinite. We take any such w and let $v_{n+1} = w$.

5. Exercise. From Raymond M. Smullyan, "Trees and ball games", *Annals of the New York Academy of Sciences* **321**(1), 86–90, 1979.

   > We consider the following one-man game: We have an infinite supply of (pool) balls numbered 1, an infinite supply of balls numbered 2, and for each positive integer n, we have infinitely may balls numbered n. We shall sometimes refer to the number on a ball as the *rank* of the ball. We shall imagine all these infinitely many balls as lying on an (infinite) floor. On a table is lying a box with infinite capacity. In this box is lying a finite number of pool balls (each ball in the box has a number, just like the balls on the floor). Now, the rule of the game is this: At any stage, the player may remove from the box any ball and then replace it with any finite number of balls of lower rank. For example, he may throw out a ball of rank 57 and replace it with a billion balls of rank 56, or rank 48, or some of one rank and some of another, providing all the ranks are less than 57. If the player throws out a ball of rank 1, he can't replace it with anything. If ever the box becomes empty, the player loses.
   >
   > The problem is whether the player must eventually lose, or whether (with sufficient ingenuity) he can keep the process going forever. On the one hand it might seem that if there is at least one ball of rank higher than 1, we can get as large a finite number of balls in the box as we please, hence we should be able to keep the process going forever. On the other hand, it might seem that even a single ball of a given rank n is somehow "worth more" than *any* finite number of balls of lower rank; hence every move we make is somehow "worsening" our situation. Which is really the case?

6. *Infinite tableaux.* We say that a finite tableau $\mathcal{T}_2$ for S is an *immediate extension* of a finite tableau $\mathcal{T}_1$ for S if one of the following holds:

   - $\mathcal{T}_2$ results from an application of a tableau expansion rule to $\mathcal{T}_1$.
   - $\mathcal{T}_2$ results from adjoining a node labeled by some $Y \in S$ at the end of some path in $\mathcal{T}_1$.

   If $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ is an infinite sequence of finite tableaux for S such that

   - $\mathcal{T}_0$ is a tableau with just one node, and
   - for every n, $\mathcal{T}_{n+1}$ is an immediate extension of $\mathcal{T}_n$,

   then
   $$\bigcup_n \mathcal{T}_n$$
   is an *(infinite) tableau* for S. Here, $\bigcup_n \langle D_n, L_n, \ell_n \rangle$ is defined to be $\langle \bigcup_n D_n, \bigcup_n L_n, \bigcup_n \ell_n \rangle$. Note that $\bigcup_n \ell_n$ is a well-defined function from $\bigcup_n D_n$ to $\bigcup_n L_n$.

7. The definition of a (finite or infinite) path *obeying* a tableau expansion rule is as before. The notions of a *finished path* and of a *finished tableau* apply to infinite tableaux with no change.

8. We say that a node labeled by X in a tableau is *fulfilled* iff for every open path P that goes through the node, the following hold:

   - if $\dfrac{X}{Y}$ is an instance of a tableau expansion rule, Y is on P,
   - if $\dfrac{X}{\begin{array}{c}Y_1\\ Y_2\end{array}}$ is an instance of a tableau expansion rule, both Y_1 and Y_2 are on P,
   - if $\dfrac{X}{Y \mid Z}$ is an instance of a tableau expansion rule, either Y or Z is on P, and
   - if $\dfrac{X}{\begin{array}{c|c}Y_1 & Z_1\\ Y_2 & Z_2\end{array}}$ is an instance of a tableau expansion rule, either Y_1 and Y_2 are on P, or else Z_1 and Z_2 are on P.

   It is easy to see that, given a finite tableau $\mathcal{T}$ and some unfulfilled nodes $v_1, \ldots, v_k$ of $\mathcal{T}$, we can find a finite tableau $\mathcal{T}'$ that extends $\mathcal{T}$ where $v_1, \ldots, v_k$ are fulfilled. (Just apply suitable tableau expansion rules to all open paths that go through $v_1, \ldots, v_k$, in turn.)

9. Lemma. For every (countably infinite)[1] set S of signed formulas, there is a finished tableau for S. (Note that if S is infinite, any finished tableau for S must be either closed or infinite.)

   Proof. Let $X_0, X_1, X_2, \ldots$ be an enumeration of the elements of S. We recursively define finite tableaux $\mathcal{T}_n$ for S, for all natural numbers n. Let $\mathcal{T}_0$ be a tableau with just one node, labeled by X_0. Now assume that $\mathcal{T}_n$ has been defined. We take all the nodes of $\mathcal{T}_n$ of level n that are not yet fulfilled, and extend $\mathcal{T}_n$ to $\mathcal{T}'_n$ by fulfilling those nodes. Then we adjoin a new node labeled by X_{n+1} at the end of every open path in $\mathcal{T}'_n$. The result is $\mathcal{T}_{n+1}$. Having defined an infinite sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of finite tableaux for S, we let $\mathcal{T} = \bigcup_n \mathcal{T}_n$. It is easy to see that $\mathcal{T}$ is a tableau, every node of $\mathcal{T}$ is fulfilled, and for all n, X_n is on every open path of $\mathcal{T}$. Therefore, $\mathcal{T}$ is a finished tableau for S.

10. Lemma. If there is a closed tableau for S, there is a finite closed tableau for S.

    Proof. Let $\mathcal{T}$ be a closed tableau for S. If $\mathcal{T}$ is finite, we are done, so assume that $\mathcal{T}$ is infinite. Let $\mathcal{T}_n$ be as in the definition of infinite tableaux, so that $\mathcal{T} = \bigcup_n \mathcal{T}_n$ and each $\mathcal{T}_{n+1}$ is a tableau for S extending $\mathcal{T}_n$. We call a node v of $\mathcal{T}$ *contradictory* if for some A, both $T\,A$ and $F\,A$ appear at nodes above v (not including the node v itself),

[1] Since we assumed that there are countably many propositional variables, there are only countably many signed formulas.

on a path going through v. Let $\mathcal{T}'$ be the result of removing all contradictory nodes from $\mathcal{T}$.

Claim 1. *The set of nodes of* $\mathcal{T}'$ *is a tree domain.* It is easy to see that the parent of a non-contradictory node is non-contradictory, and any sibling of a non-contradictory node is non-contradictory.

Claim 2. $\mathcal{T}'$ *is finite.* $\mathcal{T}'$ is a finitely branching tree because $\mathcal{T}$ is. By Kőnig's Lemma, it suffices to show that $\mathcal{T}'$ has no infinite path. Suppose that $\mathcal{T}'$ has an infinite path ρ. Then ρ must be a path in $\mathcal{T}$. Since every node on ρ is non-contradictory, there is no formula A such that both $T\,A$ and $F\,A$ appear on ρ. So ρ is an open path in $\mathcal{T}$, contradicting the assumption that $\mathcal{T}$ is a closed tableau.

Claim 3. *For every path* ρ *in* $\mathcal{T}'$*, there is a formula* A *such that both* $T\,A$ *and* $F\,A$ *appear on* ρ*.* Let ρ be a path in $\mathcal{T}'$ ending in leaf v. If v is a leaf in $\mathcal{T}$, ρ is a path in $\mathcal{T}$. Since every path of $\mathcal{T}$ is closed, there must be some A such that both $T\,A$ and $F\,A$ appear on ρ. Suppose that v has a child u in $\mathcal{T}$. Since u is not a node of $\mathcal{T}'$, u is contradictory. So there must be some A such that both $T\,A$ and $F\,A$ appear on ρ.

Since $\mathcal{T}'$ is finite and $\mathcal{T} = \bigcup_n \mathcal{T}_n$, there exists some i such that $\mathcal{T}_i$ contains all nodes of $\mathcal{T}'$. Suppose that $\mathcal{T}_i$ has an open path ρ ending in leaf v. Then v is a non-contradictory node in $\mathcal{T}$, so v is a node in $\mathcal{T}'$. This implies that ρ is a path in $\mathcal{T}'$, contradicting Claim 3. Therefore, $\mathcal{T}_i$ is a finite closed tableau for S.

Note: Since some tableau expansion rules simultaneously add two nodes to one path, $\mathcal{T}'$ *may not be a tableau, although it is very close to being one. Smullyan actually allows those tableau expansion rules to add just one node, so according to his definition,* $\mathcal{T}'$ *would be a tableau.*

11. Exercise. Show that the finished tableau constructed by the procedure in Lemma (9) is either an open tableau or a finite closed tableau.

12. Completeness Theorem (for the general case). Every (finite or countably infinite) consistent set of signed formulas is satisfiable.

    Proof. Suppose that $S = \{ X_i \mid i \in \mathbb{N} \}$ is a consistent set of signed formulas. There must be a finished tableau $\mathcal{T}$ for S. Since there is no finite closed tableau for S, $\mathcal{T}$ must be an open tableau for S. By the Lemma for the Completeness Theorem (26 of "Tableaux for propositional logic"), S is satisfiable.

13. Compactness Theorem. Let S be a countably infinite set of signed formulas. If every finite subset of S is satisfiable, then S is satisfiable.

    Proof. Suppose that S is unsatisfiable. By the Completeness Theorem, S is inconsistent, i.e., there is a finite closed tableau $\mathcal{T}$ for S. Since $\mathcal{T}$ is finite, there is some finite subset S_0 of S such that $\mathcal{T}$ is a tableau for S_0. By the Soundness Theorem, S_0 is unsatisfiable.

14. A *graph* is a structure $G = (V, E)$, where V is a set of *vertices* and E is a set of unordered pairs of distinct elements of V (i.e., $E \subseteq \{ \{u, v\} \mid u, v \in V, u \neq v \}$. If $\{u, v\} \in E$, u and v are said to be *adjacent*. Let $G = (V, E)$ be a graph and let k be a positive integer. A *k-coloring* of G is a function $f\colon V \to \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever u and v are adjacent. (The condition $f(v) = i$ means "vertex v gets color i".) A graph G is said to be *k-colorable* iff there exists a k-coloring of G.

15. Exercise. Let $G = (V, E)$ be a graph and k be a positive integer. For each vertex v and each $i = 1, \ldots, k$, let p_{vi} be a propositional variable. We interpret p_{vi} to mean "vertex v gets color i". Define $C_k(G)$ to be the set consisting of the following formulas:

    - $p_{v1} \vee \cdots \vee p_{vk}$ for each $v \in V$,
    - $\neg(p_{vi} \wedge p_{vj})$ for each $v \in V$ and $1 \leq i < j \leq k$, and
    - $\neg(p_{ui} \wedge p_{vi})$ for each $\{u, v\} \in E$ and $1 \leq i \leq k$.

    (a) Show that there is a one-to-one correspondence between k-colorings of G and assignments satisfying $C_k(G)$.
    (b) Show that G is k-colorable if and only if $C_k(G)$ is satisfiable.
    (c) Show that G is k-colorable if and only if each finite subgraph of G is k-colorable. (This is known as the *de Bruijn-Erdős theorem.*)