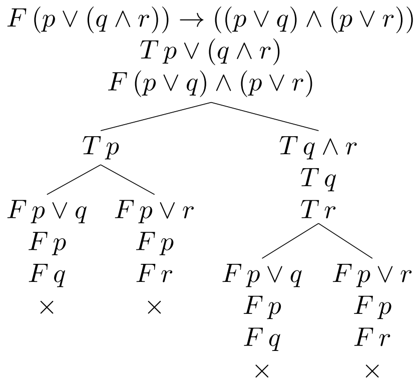
Tableaux for Propositional Logic

1. Signed formulas: $T\,A$, $F\,A$

$$v_M(T\,A) = v_M(A)$$

$$v_M(F\,A) = \begin{cases} t & \text{if } v_M(A) = f, \\ f & \text{if } v_M(A) = t. \end{cases}$$

2. A *(signed) tableau* is a certain kind of binary, labeled ordered tree where each node is labeled by a signed formula. A *path* in a tableau is a sequence of nodes starting from the root node and ending in a leaf node (i.e., node without children).
3. Example: Show $(p \vee (q \wedge r)) \to ((p \vee q) \wedge (p \vee r))$ is a tautology.



4. Tableau expansion rules:

$$\frac{T\,\neg A}{F\,A} \qquad \frac{F\,\neg A}{T\,A} \qquad \frac{T\,A \to B}{F\,A \mid T\,B} \qquad \frac{F\,A \to B}{\begin{array}{c} T\,A \\ F\,B \end{array}}$$

$$\frac{T\,A \wedge B}{\begin{array}{c} T\,A \\ T\,B \end{array}} \qquad \frac{F\,A \wedge B}{F\,A \mid F\,B} \qquad \frac{T\,A \leftrightarrow B}{\begin{array}{c|c} T\,A & F\,A \\ T\,B & F\,B \end{array}} \qquad \frac{F\,A \leftrightarrow B}{\begin{array}{c|c} T\,A & F\,A \\ F\,B & T\,B \end{array}}$$

$$\frac{T\,A \vee B}{T\,A \mid T\,B} \qquad \frac{F\,A \vee B}{\begin{array}{c} F\,A \\ F\,B \end{array}}$$

5. Meanings of tableau expansion rules:

$\dfrac{X}{Y}$: adjoin a new node labeled by Y at the end of a path passing through a node labeled by X

$\dfrac{X}{\begin{array}{c} Y_1 \\ Y_2 \end{array}}$: adjoin a new node labeled by Y_1, then a new node labeled by Y_2, at the end of a path passing through a node labeled by X

$\dfrac{X}{Y \mid Z}$: adjoin a new node labeled by Y and a new node labeled by Z at the end of a path passing through a node labeled by X (the two new nodes become the two children of the endpoint of the old path, splitting the path into two)

$\dfrac{X}{\begin{array}{c|c} Y_1 & Z_1 \\ Y_2 & Z_2 \end{array}}$: adjoin a new node labeled by Y_1 and a new node labeled by Z_1 at the end of a path passing through a node labeled by X (the two new nodes become the two children of the endpoint of the old path, splitting the path into two), then a new node labeled by Y_2 at the end of the first path, and a new node labeled by Z_2 at the end of the second path

6. Tableaux for a signed formula X:
   - The tree with just one node labeled by X is a tableau for X.
   - If $\mathcal{T}_1$ is a tableau for X, and $\mathcal{T}_2$ results from an application of a tableau expansion rule to $\mathcal{T}_1$, then $\mathcal{T}_2$ is a tableau for X.
7. A path is *closed* iff it contains both $T\,A$ and $F\,A$ for some A; otherwise *open*. A tableau is *closed* iff all its paths are closed; otherwise *open*.
8. A path ρ *obeys* tableau expansion rule
   - $\dfrac{X}{Y}$ if whenever a signed formula of the form X is on ρ, Y is also on ρ.
   - $\dfrac{X}{\begin{array}{c}Y_1\\Y_2\end{array}}$ if whenever a signed formula of the form X is on ρ, both Y_1 and Y_2 are on ρ.
   - $\dfrac{X}{Y \mid Z}$ if whenever a signed formula of the form X is on ρ, either Y or Z (or both) is on ρ.
   - $\dfrac{X}{\begin{array}{c|c}Y_1 & Z_1\\Y_2 & Z_2\end{array}}$ if whenever a signed formula of the form X is on ρ, either Y_1 and Y_2 are on ρ, or else Z_1 and Z_2 are on ρ.
9. A path ρ in a tableau for X is *finished* (*complete/replete*) iff it obeys all the tableau expansion rules. A tableau for X is *finished* (*completed/replete*) if all its open paths are finished.
10. There is an algorithm that, given a signed formula X, finds a finite finished tableau for X.[1]
11. Examples. $F\,((p \to q) \wedge (q \to r)) \to (p \to r) \qquad F\,\neg((p \vee q) \to (p \wedge q))$
12. Tableaux for a set S of signed formulas:
   - If $X \in S$, then the tree with just one node labeled by X is a tableau for S.
   - If $\mathcal{T}_1$ is a tableau for S and $\mathcal{T}_2$ results from an application of a tableau expansion rule to $\mathcal{T}_1$, then $\mathcal{T}_2$ is a tableau for S.
   - If $\mathcal{T}_1$ is a tableau for S and $\mathcal{T}_2$ results from adjoining a node labeled by some $Y \in S$ at the end of some path in $\mathcal{T}_1$, then $\mathcal{T}_2$ is a tableau for S.

   (If S is finite, all its members can be introduced before applying any tableau expansion rules.)
13. A path ρ in a tableau for S is *finished* iff ρ obeys all the tableau expansion rules and all signed formulas in S are on ρ. A tableau for S is *finished* iff all its open paths are finished.
14. There is an algorithm that, given a *finite* set S of signed formulas, finds a finite finished tableau for S.
15. A signed formula X is *consistent* iff there is no finite closed tableau for X. A (finite or infinite) set S of signed formulas is *consistent* iff there is no finite closed tableau for S. (Note that a finite closed tableau for S must be a closed tableau for some finite subset of S.)
16. Soundness Theorem. Every satisfiable signed formula is consistent. Every satisfiable set of signed formulas is consistent.
17. A *tableau proof* of an unsigned formula A is a finite closed tableau for $F\,A$. An unsigned formula A is *provable* iff there is a proof for A.
18. A *tableau proof* of an unsigned formula A from a set S of unsigned formulas is a finite closed tableau for $\{\,T\,B \mid B \in S\,\} \cup \{F\,A\}$. We say A is *deducible* from S if there is a tableau proof of A from S.

[1] As defined above, all tableaux are finite. However, we will introduce a way of building infinite tableaux later.

19. Exercise. Find a tableau proof of

$$((((p \to q) \to (\neg r \to \neg s)) \to r) \to t) \to ((t \to p) \to (s \to p)).$$

20. Exercise. Find a tableau proof of $p \wedge q \wedge r \wedge s$ from

$$\{p \to q, q \to r, r \to s, s \to p, p \vee q \vee r \vee s\}.$$

21. Soundness Theorem (an alternative formulation for the single formula case). Every provable formula is a tautology.

22. Soundness Theorem (an alternative formulation). Let A be an unsigned formula and S be a set of unsigned formulas. If A is provable from S, then A is a truth-functional consequence of S.

23. Soundness Theorem (restated). If a set S of signed formulas is satisfiable, every finite tableau for S is open.

    Proof. Let M be an assignment such that $v_M(X) = t$ for all $X \in S$. Let $\mathcal{T}$ be a finite tableau for S. We show by induction on the construction of $\mathcal{T}$ that $\mathcal{T}$ has a path ρ such that $v_M(Y) = t$ for all Y on ρ. This obviously implies that ρ is open.

    Induction Basis. $\mathcal{T}$ consists of just one node labeled by some $X \in S$. Then $v_M(X) = t$ by assumption.

    Induction Step.

    Case 1. $\mathcal{T}$ results from an application of a tableau expansion rule to a path ρ' in $\mathcal{T}'$. By induction hypothesis, $\mathcal{T}'$ has a path ρ such that $v_M(Y) = t$ for all Y on ρ. Case 1a. $\rho \neq \rho'$. Then ρ is still a path in $\mathcal{T}$, so the claim holds for $\mathcal{T}$ with respect to ρ. Case 1b. $\rho = \rho'$. Then ρ is extended to one or two paths in $\mathcal{T}$. At least one of these paths must satisfy the condition in the claim, as can be easily verified by examining each tableau expansion rule.

    Case 2. $\mathcal{T}$ results from adjoining some $X \in S$ to some path ρ' in $\mathcal{T}'$. By induction hypothesis, $\mathcal{T}'$ has a path ρ such that $v_M(Y) = t$ for all Y on ρ. Case 2a. $\rho \neq \rho'$. Then ρ is still a path in $\mathcal{T}$, so the claim holds for $\mathcal{T}$ with respect to ρ. Case 2b. $\rho = \rho'$. Then ρ is extended to a path in $\mathcal{T}$ with an additional node labeled by some $X \in S$. But $v_M(X) = t$ by assumption, so all signed formulas on this new path are true under M.

24. Completeness Theorem. Every consistent signed formula X is satisfiable. Every consistent set S of signed formulas is satisfiable.

    We first prove the Completeness Theorem for the case where S is a finite set.

25. Completeness Theorem (an alternative formulation for the single formula case). If a formula A is a tautology, then A is provable.

26. Lemma for the Completeness Theorem. If there is a finished open tableau for a signed formula X, then X is satisfiable. If there is a finished open tableau for a set S of signed formulas, then S is satisfiable.

27. A set S of signed formulas is a *Hintikka set* iff

    - There is no propositional variable p such that both $T\,p$ and $F\,p$ are in S, and
    - S obeys the tableau expansion rules.[2]

28. Hintikka's Lemma. If a set S of signed formulas is a Hintikka set, then S is satisfiable.

    Proof. Define an assignment M by

$$M(p) = \begin{cases} t & \text{if } T\,p \in S, \\ f & \text{otherwise.} \end{cases}$$

    We show by induction that for all formulas A, $T\,A \in S$ implies $v_M(T\,A) = t$ and $F\,A \in S$ implies $v_M(F\,A) = t$.

[2] The definition of what it means for a *set* of signed formulas to obey a tableau expansion rule is almost identical to the definition of what it means for a *path* in a tableau to obey the rule.

Induction Basis. A is some propositional variable p. If $T\,p \in S$, then $v_M(p) = t$ by the definition of M. If $F\,p \in S$, then since S is a Hintikka set, $T\,p \notin S$. So $v_M(p) = f$ and $v_M(F\,p) = t$ by the definition of M.

Induction Step. Case 1. $A = \neg B$. If $T\,A \in S$, then since S is a Hintikka set, $F\,B \in S$. By induction hypothesis, $v_M(F\,B) = t$, so $v_M(T\,A) = t$. If $F\,A \in S$, then since S is a Hintikka set, $T\,B \in S$. By induction hypothesis, $v_M(T\,B) = t$, so $v_M(F\,A) = t$.

Case 2. $A = B \wedge C$. If $T\,A \in S$, then since S is a Hintikka set, $T\,B, T\,C \in S$. By induction hypothesis, $v_M(T\,B) = v_M(T\,C) = t$, so $v_M(T\,A) = t$. If $F\,A \in S$, then since S is a Hintikka set, either $F\,B$ or $F\,C$ is in S. By induction hypothesis, either $v_M(F\,B) = t$ or $v_M(F\,C) = t$. In either case, $v_M(F\,A) = t$.

Cases 3–5. Similar.

29. Proof of the Lemma for the Completeness Theorem. Let $\mathcal{T}$ be a finished open tableau for S. Then $\mathcal{T}$ has an open path ρ (which is finished). Since ρ is open and obeys all the tableau expansion rules, the set of signed formulas on ρ is a Hintikka set, and is thus satisfiable. Since all signed formulas in S appear on ρ, this means that S is satisfiable.
30. Proof of the Completeness Theorem (for the single formula case). Suppose that A is not provable. This means that there is no finite closed tableau for $F\,A$. But $F\,A$ always has a finite finished tableau, which must be open. By the Lemma for the Completeness Theorem, $F\,A$ is satisfiable, so A is not a tautology.
31. The Completeness Theorem for the finite set case is left as an exercise.