Propositional Logic

1. Language of propositional logic

   - propositional variables: $p_1, p_2, p_3, \ldots \quad p, q, r, s, \ldots$
   - inductive definition of *formulas*:

     (a) If p is a propositional variable, p is a formula.
     (b) If A is a formula, $\neg A$ is a formula.
     (c) If A and B are formulas, then

     $$(A \wedge B) \quad (A \vee B) \quad (A \to B) \quad (A \leftrightarrow B)$$

     are formulas.

   Let $\mathbb{P}$ denote the set of propositional variables, and $\mathbb{F}$ denote the set of formulas.

2. Example: $\big((((p_1 \to p_2) \wedge (p_2 \vee p_3)) \to (p_1 \vee p_3)) \to \neg(p_2 \vee p_4)\big)$
3. Proof by induction. If a set $\mathcal{X}$ satisfies the following conditions, then $\mathbb{F} \subseteq \mathcal{X}$.

   - $\mathbb{P} \subseteq \mathcal{X}$
   - $A \in \mathcal{X}$ implies $\neg A \in \mathcal{X}$
   - $A \in \mathcal{X}$ and $B \in \mathcal{X}$ imply $(A \, b \, B) \in \mathcal{X}$ for each $b \in \{\wedge, \vee, \to, \leftrightarrow\}$.

4. Exercise. Prove that for every formula $A \in \mathbb{F}$, the number of occurrences of propositional variables in A is the number of occurrences of '(' (open parenthesis) plus 1.
5. Convention: Omit outermost pair of parentheses: $(((p_1 \to p_2) \wedge (p_2 \vee p_3)) \to (p_1 \vee p_3)) \to \neg(p_2 \vee p_4)$
6. Unique Readability. For every formula A, exactly one of the following holds:

   (a) $A = p$ for some propositional variable p.
   (b) $A = \neg B$ for some formula B.
   (c) $A = (B_1 \, b \, B_2)$ for some formulas B_1, B_2 and $b \in \{\wedge, \vee, \to, \leftrightarrow\}$.

   Moreover, in case $A = \neg B$, the choice of B is unique, and in case $A = (B_1 \, b \, B_2)$, the choice of B_1, B_2, b is unique. The connective $\neg$ (in case of (b)) or b (in case of (c)) is called the *principal connective* of A.

7. Recursive definition. Let $\mathcal{S}$ be some set, and $g \colon \mathbb{P} \to \mathcal{S}$, $h_\neg \colon \mathcal{S} \to \mathcal{S}$, $h_b \colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ ($b \in \{\wedge, \vee, \to, \leftrightarrow\}$) be some functions. The following set of equations defines a function $f \colon \mathbb{F} \to \mathcal{S}$.

   - $f(p) = g(p)$ for each $p \in \mathbb{P}$,
   - $f(\neg A) = h_\neg(f(A))$ for each $A \in \mathbb{F}$,
   - $f(A \, b \, B) = h_b(f(A), f(B))$ for each $A, B \in \mathbb{F}$ and $b \in \{\wedge, \vee, \to, \leftrightarrow\}$.

8. Example. The *height* of $A \in \mathbb{F}$ is defined by

   $$\begin{aligned} h(p) &= 0 && \text{for } p \in \mathbb{P}, \\ h(\neg A) &= h(A) + 1 && \text{for } A \in \mathbb{F}, \\ h(A \, b \, B) &= \max(h(A), h(B)) + 1 && \text{for } A, B \in \mathbb{F} \text{ and } b \in \{\wedge, \vee, \to, \leftrightarrow\}. \end{aligned}$$

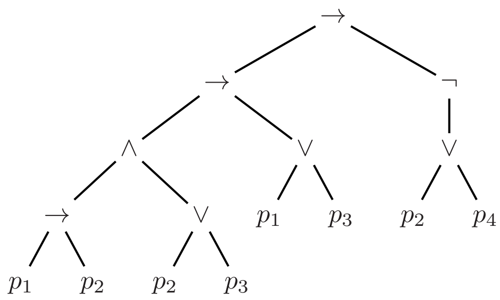
9. *Subformulas:*

   $$\begin{aligned} Sub(p) &= \{p\} \\ Sub(\neg A) &= \{\neg A\} \cup Sub(A) \\ Sub(A \, b \, B) &= \{A \, b \, B\} \cup Sub(A) \cup Sub(B) \end{aligned}$$

10. Proposition. If $A \in Sub(B)$, then $Sub(A) \subseteq Sub(B)$.
11. *Formation tree* for A: a labeled ordered binary (i.e., at most binary-branching) tree such that

- each node is labeled by a subformula of A,
- the root is labeled by A,
- each node labeled by $\neg B$ has a node labeled by B as its only child,
- each node labeled by $B \, b \, C$ has a node labeled by B and a node labeled by C, in this order, as its only children,
- each node labeled by p is a leaf.

12. Example: In abbreviated notation,



13. *Truth values:* t, f
14. *Truth assignment:*

$$M \colon \mathbb{P} \to \{t, f\}$$

15. If M is a truth assignment, extend M to *valuation*

$$v_M \colon \mathbb{F} \to \{t, f\}$$

by

$$v_M(p) = M(p)$$
$$v_M(\neg A) = \begin{cases} t & \text{if } v_M(A) = f, \\ f & \text{if } v_M(A) = t, \end{cases}$$
$$v_M(A \wedge B) = \begin{cases} t & \text{if } v_M(A) = v_M(B) = t, \\ f & \text{otherwise,} \end{cases}$$
$$v_M(A \vee B) = \begin{cases} t & \text{if at least one of } v_M(A) = t \text{ and } v_M(B) = t \text{ holds,} \\ f & \text{otherwise,} \end{cases}$$
$$v_M(A \to B) = \begin{cases} t & \text{if at least one of } v_M(A) = f \text{ and } v_M(B) = t \text{ holds,} \\ f & \text{otherwise,} \end{cases}$$
$$v_M(A \leftrightarrow B) = \begin{cases} t & \text{if } v_M(A) = v_M(B), \\ f & \text{otherwise.} \end{cases}$$

16. Proposition. If M_1 and M_2 agree on the propositional variables in $Sub(A)$, then $v_{M_1}(A) = v_{M_2}(A)$.
17. *Truth table* for A

q_1	$\dots$	q_n	A
t	$\dots$	t	
	$\vdots$		
b_1	$\dots$	b_n	b
	$\vdots$		
f	$\dots$	f	

$q_1, \dots, q_n$: propositional variables in $Sub(A)$. Each row expresses $v_M(A) = b$ for all M such that $M(q_1) = b_1, \dots, M(q_n) = b_n$.

18. Example.

p	q	r	$(p \wedge \neg q) \to \neg(p \vee r)$
t	t	t	t
t	t	f	t
t	f	t	f
t	f	f	f
f	t	t	t
f	t	f	t
f	f	t	t
f	f	f	t

19. Let M be an assignment, A be a formula, and S be a set of formulas.

   - A is *true under* M iff $v_M(A) = t$, *false under* M iff $v_M(A) = f$.
   - M *satisfies* A iff A is true under M.
   - A is satisfiable iff at least one assignment satisfies A.
   - M *satisfies* S iff M satisfies all A in S.
   - A is *truth-functionally valid* (or is a *tautology*) iff A is true under all assignments.
   - A is a *truth-functional consequence of* S iff all assignments that satisfy S satisfy A.
   - A is *truth-functionally equivalent to* B iff A and B are true under the same assignments.

20. Proposition.

   (a) A is a tautology iff $\neg A$ is not satisfiable.
   (b) B is a truth-functional consequence of $\{A\}$ iff $A \to B$ is a tautology.
   (c) A is truth-functionally equivalent to B iff $A \leftrightarrow B$ is a tautology.

21. Proposition. Let A be a formula and let $p_1, \ldots, p_n$ be the list of all propositional variables in A. If A is a tautology, then so is $A[B_1/p_1, \ldots, B_n/p_n]$ for any formulas $B_1, \ldots, B_n$, where $A[B_1/p_1, \ldots, B_n/p_n]$ is obtained from A by replacing p_i with B_i for $i = 1, \ldots, n$.

22. Example.

   (a) $((A \to B) \to A) \to A$ is a tautology (for all A, B). (*Peirce's Law*)
   (b) $(A \wedge B) \to C$ is truth-functionally equivalent to $A \to (B \to C)$.

23. Write $A \equiv B$ for "A is truth-functionally equivalent to B".

24. Proposition. If $A_1 \equiv A_2$, then

   (a) $\neg A_1 \equiv \neg A_2$
   (b) $A_1 \wedge B \equiv A_2 \wedge B$
   (c) $B \wedge A_1 \equiv B \wedge A_2$
   (d) $A_1 \vee B \equiv A_2 \vee B$
   (e) $B \vee A_1 \equiv B \vee A_2$
   (f) $A_1 \to B \equiv A_2 \to B$
   (g) $B \to A_1 \equiv B \to A_2$
   (h) $A_1 \leftrightarrow B \equiv A_2 \leftrightarrow B$
   (i) $B \leftrightarrow A_1 \equiv B \leftrightarrow A_2$

25. Proposition. If $A_1 \equiv A_2$, then $B \equiv B'$, where B' is the result of replacing one or more occurrences of A_1 in B by A_2.

26. Convention: Write

$$A_1 \vee \cdots \vee A_n \quad \text{for} \quad \big(\ldots(A_1 \vee A_2) \vee \ldots\big) \vee A_n$$
$$B_1 \wedge \cdots \wedge B_n \quad \text{for} \quad \big(\ldots(B_1 \wedge B_2) \wedge \ldots\big) \wedge B_n$$

   This is justified by $A \vee (B \vee C) \equiv (A \vee B) \vee C$ and $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$.

27. A *literal* is a propositional variable p or its negation $\neg p$.
28. A is in *disjunctive normal form* if it is of the form $A_1 \vee \cdots \vee A_m$ where each A_i is of the form $l_1 \wedge \cdots \wedge l_n$ where each l_j is a literal.
29. Proposition. Every formula is truth-functionally equivalent to one in disjunctive normal form.
30. Example. $(p \to q) \to r$ has the following truth table:

p	q	r	$(p \to q) \to r$
t	t	t	t
t	t	f	f
t	f	t	t
t	f	f	t
f	t	t	t
f	t	f	f
f	f	t	t
f	f	f	f

Therefore, $(p \to q) \to r$ is truth-functionally equivalent to

$$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

31. Exercise. Give an efficient algorithm for solving the following problem:

    DNF SATISFIABILITY
    INSTANCE: A propositional formula A in disjunctive normal form.
    QUESTION: Is A satisfiable?

32. Exercise. Consider the following puzzle:

    Imagine an island inhabited by two types of people, Liars and Truth-Tellers. If a person X is a Liar, everything X says is false, while if X is a Truth-Teller, everything X says is true. Suppose H and K are two inhabitants of this island. Suppose H said: "At least one of H and K is a Liar." Who of H and K is a Liar?

    The solution to this puzzle is as follows. Assume H is a Liar. Then what H said is false, so neither H nor K is a Liar. This is a contradiction. So H is not a Liar, and what H said is true. Since H is not a Liar, K must be a Liar.

    This puzzle can be more systematically solved using truth tables. Let p stand for "H is a Liar", and q for "K is a Liar." Then what H said is $p \vee q$. The assumption of this puzzle is that p and what H said have opposite truth values. See the truth table for $p \vee q$:

p	q	$p \vee q$
t	t	t
t	f	t
f	t	t
f	f	f

    The only row in which the truth values of p and $p \vee q$ differ is the third row. So p is false and q is true.

    (a) Under the same assumptions, suppose H said: "Either H is a Liar or K is not a Liar." What can you conclude from H's statement?
    (b) Under the same assumptions, suppose H said something and from H's statement it followed that K is not a Liar, but no conclusion was drawn as to whether or not H is a Liar. What did H say?
    (c) Let A be what H said and B be what can be concluded from the fact that H said A. What is the relation between A and B?